Version 1.1 November 2006

AQA GCE Mathematics Specification (6360)

MS03

Proofs of Mean and Variance of Binomial and Poisson Distributions

Proofs of Mean and Variance of Binomial and Poisson Distributions

1 Binomial

1.1 Mean

Mean = E(X) = $\mu = \sum x_i p_i$ where $\sum p_i = 1$	Definition
$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$	Value is zero at $x = 0$
$= \sum_{x=1}^{n} x \dfrac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$	Definition of $\binom{n}{x}$
$= np \sum_{x=1}^{n-1} \dfrac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}$	Take out factor of np Upper limit of summation is then $(n-1)$ Cancel x as $x! = x(x-1)!$
$= np \sum_{y=0}^{m} \dfrac{m!}{y!(m-y)!} p^m (1-p)^{m-y}$	Substitution of $y = x - 1$ Substitution of $m = n - 1$ Noting $(n-x) = (m-y)$
$= np \times \sum p_i = np \times 1 = \underline{np}$	Sum of all binomial terms = Sum of all probabilities = 1

1.2 Variance

Variance = $E(X^2) - (E(X))^2 = \sigma^2 = \sum x_i^2 p_i - \mu^2$ where $\sum p_i = 1$	Definition
(a) Firstly consider $E(X(X-1)) = E(X^2) - E(X) = E(X^2) - \mu$	Expansion
$= [E(X^2) - \mu^2] + \mu^2 - \mu$	Add & subtract $\mu^2 = (E(X))^2$
$= \sigma^2 + \mu^2 - \mu$	Substitution
Thus $\sigma^2 = E(X(X-1)) - \mu^2 + \mu = \underline{E(X(X-1)) - n^2p^2 + np}$	Rearranging & substitution of $\mu = np$
(b) Secondly reconsider $E(X(X-1))$	
$= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$	Definition
$= \sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$	Value is zero at $x = 0$ & 1
$= \sum_{x=2}^{n} x(x-1)\frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$	Definition of $\binom{n}{x}$
$= n(n-1)p^2 \sum_{x=2}^{n-2} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} (1-p)^{n-x}$	Take out factor of $n(n-1)p^2$ Upper limit of summation is then $(n-2)$ Cancel $x(x-1)$ as $x! = x(x-1)(x-2)!$
$= n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^m (1-p)^{m-y}$	Substitution of $y = x - 2$ Substitution of $m = n - 2$ Noting $(n - x) = (m - y)$
$= n(n-1)p^2 \times \sum p_i = n(n-1)p^2 \times 1 = \underline{n(n-1)p^2}$	Sum of all binomial terms = Sum of all probabilities = 1
(c) Finally, using parts (a) and (b)	
$\sigma^2 = E(X(X-1)) - n^2p^2 + np = n(n-1)p^2 - n^2p^2 + np$	
$= n^2p^2 - np^2 - n^2p^2 + np$	Expanding
$= np - np^2 = \underline{np(1-p)}$	Cancelling & factorising

2 Poisson

2.1 Mean

Mean = $E(X) = \mu = \sum x_i p_i$ where $\sum p_i = 1$	Definition
$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$	Value is zero at $x = 0$
$= \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!}$	Take out factor of λ Cancel x as $x! = x(x-1)!$ Substitution of $y = x - 1$
$= \lambda \times \sum p_i = \lambda \times 1 = \underline{\lambda}$	Sum of all Poisson terms = sum of all probabilities = 1

2.2 Variance

Variance = $E(X^2) - (E(X))^2 = \sigma^2 = \sum x_i^2 p_i - \mu^2$ where $\sum p_i = 1$	Definition
(a) Firstly consider $E(X(X-1)) = E(X^2) - E(X) = E(X^2) - \mu$	Expansion
$= [E(X^2) - \mu^2] + \mu^2 - \mu$	Add & subtract $\mu^2 = (E(X))^2$
$= \sigma^2 + \mu^2 - \mu$	Substitution
Thus $\sigma^2 = E(X(X-1)) - \mu^2 + \mu = \underline{E(X(X-1)) - \lambda^2 + \lambda}$	Rearranging & substitution of $\mu = \lambda$
(b) Secondly reconsider $E(X(X-1))$	
$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!}$	Definition Value is zero at $x = 0$ & 1
$= \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!}$	Take out factor of λ^2 Cancel $x(x-1)$ as $x! = x(x-1)(x-2)!$ Substitution of $y = x - 2$
$= \lambda^2 \times \sum p_i = \lambda^2 \times 1 = \underline{\lambda^2}$	Sum of all Poisson terms = Sum of all probabilities = 1
(c) Finally, using parts (a) and (b)	
$\sigma^2 = E(X(X-1)) - \lambda^2 + \lambda = \lambda^2 - \lambda^2 + \lambda$	Expanding
$= \underline{\lambda}$	Cancelling

3 Some Alternative Approaches

3.1 Binomial

The following results may be quoted/used in the proofs.

Probability generating function $G(t) = E(t^X) = (q + pt)^n$ where $q = 1 - p$

Moment generating function $M(t) = E(e^{tX}) = (q + pe^t)^n$ where $q = 1 - p$

The following properties may then be used to find the mean and variance.

$$\mu = \left.\frac{dG(t)}{dt}\right|_{t=1} \quad \text{or} \quad \mu = \left.\frac{dM(t)}{dt}\right|_{t=0}$$

$$\sigma^2 = \left.\frac{d^2G(t)}{d^2t}\right|_{t=1} + \mu - \mu^2 \quad \text{or} \quad \sigma^2 = \left.\frac{d^2M(t)}{d^2t}\right|_{t=0} - \mu^2$$

3.2 Poisson

The following results may be quoted/used in the proofs.

Probability generating function $G(t) = E(t^X) = e^{\lambda t - \lambda}$

Moment generating function $M(t) = E(e^{tX}) = e^{\lambda e^t - \lambda}$

The following properties may then be used to find the mean and variance.

$$\mu = \left.\frac{dG(t)}{dt}\right|_{t=1} \quad \text{or} \quad \mu = \left.\frac{dM(t)}{dt}\right|_{t=0}$$

$$\sigma^2 = \left.\frac{d^2G(t)}{d^2t}\right|_{t=1} + \mu - \mu^2 \quad \text{or} \quad \sigma^2 = \left.\frac{d^2M(t)}{d^2t}\right|_{t=0} - \mu^2$$