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GCE Mathematics *(6360)*

Further Pure unit 4 *(MFP4)*

Textbook

Further Pure 4: Contents

Chapter 1: Matrix Algebra

This chapter introduces the idea of a matrix. When you have completed it, you will:

- know what is meant by a matrix;
- be able to add, subtract and multiply matrices;
- be able to multiply matrices by scalars;
- know what is meant by the transpose of a matrix;
- understand how matrices can represent transformations;
- know the matrices for some transformations;
- be able to calculate the matrices for combined transformations.

1.1 Matrices

Any rectangular array of numbers is called a **matrix** (the plural is **matrices)**.

For example,

$$\mathbf{M} = \begin{bmatrix} -1 & 3 \\ 3 & 5 \\ 4 & -1 \end{bmatrix}$$

is a matrix. **M** has three rows and two columns and is called a matrix of **order** $\mathbf{3 \times 2}$ or, simply, a $\mathbf{3 \times 2}$ **matrix**. Note that each row or column is itself a matrix. For example, the third row is $\begin{bmatrix} 4 & -1 \end{bmatrix}$, a 1×2 matrix.

> When giving the order of a matrix, you should always give the number of rows first, then the number of columns

Each number in a matrix is called an **element**.

Exercise 1A

In the following questions, **A** is the matrix defined by

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 & -1 \\ 1 & -1 & 1 & 4 \end{bmatrix}.$$

1. What is the order of matrix **A**?

2. Which element of **A** is in the first row and in the third column?

3. What type of matrix is the fourth column of **A**?

4. A triangle has coordinates (3, 4), (–1, 2), (2, –3). Represent this triangle by a 2×3 matrix with the coordinates forming the columns.

1.2 Matrix arithmetic

Consider four sports teams – A, B, C and D. The numbers of wins, draws and losses for all the teams after they have played each other once can be expressed in a matrix:

$$\mathbf{R} = \begin{array}{c} \begin{matrix} \text{W} & \text{D} & \text{L} \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

Team A wins 0 games, draws 2 and loses 1

The results of one further set of games, when A plays B and C plays D, can be expressed in a matrix of the same order:

$$\mathbf{S} = \begin{array}{c} \begin{matrix} \text{W} & \text{D} & \text{L} \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

The results after all the matches, when each team has played four games, can then be summarised as

$$\begin{array}{c} \begin{matrix} \text{W} & \text{D} & \text{L} \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

The element 0 of **R** plus the element 1 of **S**

The element 2 of **R** plus the element 1 of **S**

The new matrix is called the sum of **R** and **S**; i.e. $\mathbf{R}+\mathbf{S}$.

> Two matrices can be added, or subtracted, if they have the **same order**. To add, or subtract, two matrices simply add, or subtract, the corresponding elements. For example,
>
> $$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix} = \begin{bmatrix} a+g & b+h & c+i \\ d+j & e+k & f+l \end{bmatrix},$$
>
> $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Suppose that the four sports teams play four more games each and produce exactly the same results as in the first games. The total of all the results could be obtained by doubling the total of the first games:

$$2\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 4 & 2 \\ 0 & 6 & 2 \end{bmatrix}.$$

When calculating with matrices, ordinary numbers are called **scalars**. Multiplication of a matrix by a scalar, as above, is called **scalar multiplication**.

> Any matrix can be multiplied by any scalar. Simply multiply each element by the scalar. For example,
>
> $$k\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \\ ke & kf \end{bmatrix}$$

Suppose that the method of awarding points to teams is 3 for a win, 1 for a draw and 0 for a loss. The points can be expressed in a 3×1 matrix,

$$\begin{matrix} \text{W} \\ \text{D} \\ \text{L} \end{matrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

The total number of points earned by each team, when each has played eight games, can be found as follows.

$$\overset{\text{Results}}{\begin{bmatrix} 2 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 4 & 2 \\ 0 & 6 & 2 \end{bmatrix}}\begin{matrix} A \\ B \\ C \\ D \end{matrix} \quad \overset{\text{Points}}{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}\begin{matrix} \text{W} \\ \text{D} \\ \text{L} \end{matrix} \quad \begin{matrix} A \\ B \\ C \\ D \end{matrix}\begin{bmatrix} 10 \\ 14 \\ 10 \\ 6 \end{bmatrix}$$

Team A earned
$(2\times3)+(4\times1)+(2\times0)=10$ points

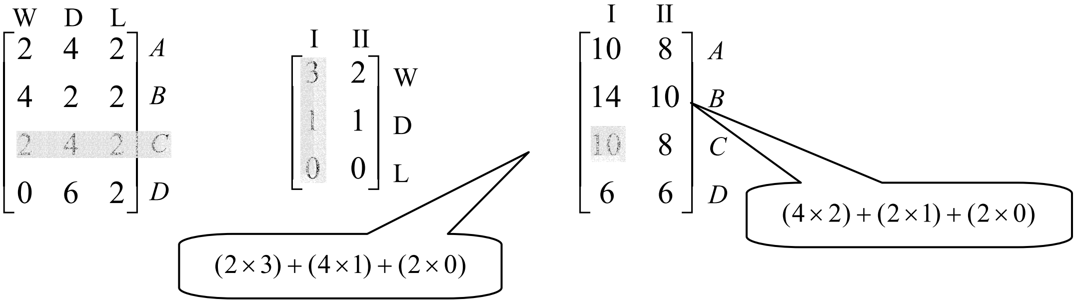
Now suppose that there are **two** methods of awarding points to teams.

- Method I : 3 points for a win; 1 point for a draw; 0 for a loss.
- Method II : 2 points for a win; 1 point for a draw; 0 for a loss.

These two possibilities can also be expressed as a matrix:

$$\begin{matrix} \text{I} & \text{II} \end{matrix}$$
$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{matrix} \text{W} \\ \text{D} \\ \text{L} \end{matrix}$$

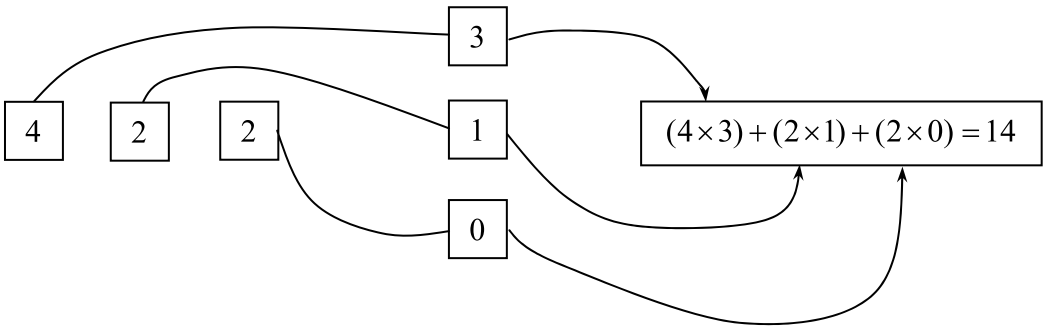
The two possibilities for the total number of points earned by each team after each team has played eight games are found as follows.



This combination of the 4×3 matrix of results and the 3×2 matrix of points is called **matrix multiplication**, i.e.

$$\begin{bmatrix} 2 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 4 & 2 \\ 0 & 6 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 14 & 10 \\ 10 & 8 \\ 6 & 6 \end{bmatrix}.$$

To multiply two matrices, the number of columns in the first matrix must equal the number of rows in the second matrix. This is because of the way that pairs of numbers are multiplied. For example, the second row of the 4×3 matrix above is multiplied by the first column of the 3×2 matrix as follows:



Consider the orders of the matrices in the example above.

left hand matrix × right hand matrix ⟶ product matrix

4×3 $\quad$ 3×2 $\quad$ 4×2

these numbers must be equal

these numbers determine the order of the product matrix

> Matrices are multiplied by multiplying the elements in a row of the first matrix by the elements in a column of the second matrix, and adding the results. For example,
>
> $$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} g & h & i & j \\ k & l & m & n \end{bmatrix} = \begin{bmatrix} ag+bk & ah+bl & ai+bm & aj+bn \\ cg+dk & ch+dl & ci+dm & cj+dn \\ eg+fk & eh+fl & ei+fm & ej+fn \end{bmatrix}$$
>
> The product **AB** can be found if the number of columns of matrix **A** is equal to the number of rows of matrix **B**. If the order of matrix **A** is $r \times s$ and **B** is $s \times t$, then the order of **AB** is $r \times t$

Example 1.2.1

If $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$, find which of the products **AB** and **BA** can be evaluated.

Solution

A is 2×2 and **B** is 3×2. Hence,

2×2 $\quad$ 3×2 (different) $\quad$ so **AB** cannot be evaluated;

3×2 $\quad$ 2×2 (equal) $\quad$ so **BA** can be evaluated and is 3×2.

$$\mathbf{BA} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 7 & -7 \\ 4 & -5 \end{bmatrix}$$

$(1 \times 1) + (-1 \times -3)$

Exercise 1B

1. If $\mathbf{A}=\begin{bmatrix}3 & 1 & -1\\1 & 2 & 0\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1 & -1\\1 & 0\\2 & 1\end{bmatrix}$, find **AB** and **BA**.

2. $\mathbf{A}=\begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}2 & 2\\-1 & 3\end{bmatrix}$.

   (a) Find $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{AB}$ and $\mathbf{BA}$.

   (b) Find $\mathbf{A}+\mathbf{B}$ and verify that $(\mathbf{A}+\mathbf{B})^2=\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$.

3. Which of the following matrices can be multiplied by themselves?

   $\mathbf{A}=\begin{bmatrix}1 & 1\\3 & 1\\1 & 2\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}3 & -1 & 1\\1 & -1 & 2\\2 & 1 & -3\end{bmatrix}$, $\mathbf{C}=\begin{bmatrix}1 & 1 & 4\end{bmatrix}$, $\mathbf{D}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

4. Let $\mathbf{A}=\begin{bmatrix}2 & 1\\3 & -1\end{bmatrix}$ and $\mathbf{I}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. Show that $\mathbf{A}^2=\mathbf{A}+5\mathbf{I}$.

5. Let $\mathbf{A}=\begin{bmatrix}1 & -1 & 2\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}1 & -1\\2 & 1\\1 & 3\end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$. Show that $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$.

6. **A** is a 1×3 matrix and **B** is a 4×2 matrix. Given that the products **AX**, **XB**, **BY** and **YA** can all be found, what are the orders of **X** and **Y**?

7. Given that $\mathbf{A}=\begin{bmatrix}2 & 1\\1 & -1\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}1 & -1 & 2\\3 & 0 & 1\end{bmatrix}$ and $\mathbf{C}=\begin{bmatrix}2 & 0 & 1\\1 & -1 & 3\end{bmatrix}$, find $\mathbf{A}(\mathbf{B}+\mathbf{C})$, **AB** and **AC**.

   What do you notice?

8. Find $\begin{bmatrix}2 & 1 & -1\\1 & 3 & 2\\3 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & 0 & -1\\1 & 3 & -1\\-2 & 1 & 0\end{bmatrix}$.

1.3 Laws of matrix arithmetic

Since the addition of two matrices simply involves the addition of corresponding elements, matrix addition is itself straightforward. In particular:

Commutativity of addition

For any two matrices which can be added (i.e. which have the same order), $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

As you have seen in Exercise 1B, matrix multiplication is not so straightforward.

Non-commutativity of multiplication

It cannot be assumed that $\mathbf{AB}=\mathbf{BA}$, even when both products exist

However, in two important respects, matrix arithmetic and ordinary arithmetic are very similar. Firstly, you can expand brackets in the usual way.

Distributive law

For any matrices of appropriate orders,

$$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC},$$
$$(\mathbf{U}+\mathbf{V})\mathbf{W}=\mathbf{UW}+\mathbf{VW}$$

Secondly, in a string of matrix multiplications, although you must **not** change the sequence of the matrices, you can multiply any adjacent pair together first. Consequently, you can write a product such as **ABC** without ambiguity.

Associative law

For any matrices of appropriate orders,

$$\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$$

A **zero matrix** is any matrix consisting entirely of zeros. Any such matrix is denoted by **0**. For example, $\mathbf{0}=\begin{bmatrix}0 & 0\end{bmatrix}$.

For all matrices of suitable orders,

$$\mathbf{A}+\mathbf{0}=\mathbf{A},\ \mathbf{0B}=\mathbf{0}\ \text{ and }\ \mathbf{C0}=\mathbf{0}$$

An **identity matrix** is any **square** matrix in which all of the elements on the leading diagonal are 1 and all other elements are zeros. Such a matrix is denoted by **I**. For example,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For all matrices of suitable orders,

$$\mathbf{AI} = \mathbf{A} \text{ and } \mathbf{IB} = \mathbf{B}$$

In particular, for any square matrix **M** of the same order as **I**, $\mathbf{MI} = \mathbf{IM} = \mathbf{M}$.

In some circumstances it can be useful to interchange the rows and columns of a matrix. This process is called **transposing** the matrix.

The transpose of a matrix **M** is obtained by interchanging the rows and columns of **M**. The transpose of **M** is denoted by $\mathbf{M}^{\mathrm{T}}$

Example 1.3.1

Given that

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 4 & -2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \\ 0 & 2 \end{bmatrix},$$

find (a) **AB**, (b) $\mathbf{A}^{\mathrm{T}}$, (c) $\mathbf{B}^{\mathrm{T}}$ and (d) $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

(e) What do you notice about **AB** and $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$?

Solution

(a) $\mathbf{AB} = \begin{bmatrix} 0 & 9 \\ 5 & -4 \\ 2 & 9 \end{bmatrix}$.

(b) $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & -2 \end{bmatrix}$.

(c) $\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix}$.

(d) $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 0 & 5 & 2 \\ 9 & -4 & 9 \end{bmatrix}$.

(e) $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ is the same as $(\mathbf{AB})^{\mathrm{T}}$.

The result noted in part (e) of Example 1.3.1 is true for **all** compatible matrices **A** and **B**.

$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

Exercise 1C

1. Given that **A** is the matrix $\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$, find a non-zero matrix **B** such that $\mathbf{AB} = \mathbf{0}$.

2. Find two non-zero matrices, **A** and **B**, such that $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$.

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 8 \\ -1 & 6 \end{bmatrix}$. Solve for **X** the equation $\mathbf{A} + 3\mathbf{X} = \mathbf{B}$.

4. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Find all matrices **B** such that $\mathbf{AB} = \mathbf{BA}$.

5. The matrices **A**, **B**, **C**, **M** and **N** are such that

$$\mathbf{M} = \mathbf{AB} \quad \text{and} \quad \mathbf{N} = \mathbf{BC}.$$

   Explain why $\mathbf{AN} = \mathbf{MC}$.

6. The matrices **A**, **B** and **C** are such that

$$\mathbf{AB} = \begin{bmatrix} 3 & 1 \\ 2 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{AC} = \begin{bmatrix} 5 & 1 \\ 7 & 5 \end{bmatrix}.$$

   Find $\mathbf{A}(\mathbf{B} - \mathbf{C})$.

7. Check the result $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ for the matrices $\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ -5 & 1 \\ 0 & 3 \end{bmatrix}$.

1.4 Matrix transformations (2-dimensional)

An important area of application of matrices is that of geometrical transformations. Consider the matrix multiplication of any 2×2 matrix by a 2×1 matrix – for example,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

You can think of the 2×1 matrices as representing the points (3, 4) and (–4, 3), respectively. The 2×2 matrix then represents a transformation which maps $P(3, 4)$ onto $P'(-4, 3)$.

Example 1.4.1

(a) A rectangle has vertices $O(0, 0)$, $A(3, 0)$, $B(3, 1)$ and $C(0, 1)$. Find the images O', A', B' and C' of these points when acted on by the transformation represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(b) Hence describe the transformation represented by **M**.

Solution

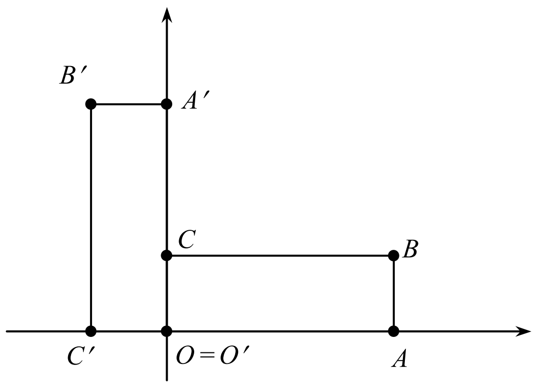
(a)

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

Write each point as a column of a single matrix

(–1, 3) is the image of (3, 1)

(b) **M** represents a rotation of 90° anticlockwise about the origin O.



> Any 2×2 matrix **M** can be thought of as representing a geometrical transformation of the points in the plane.
>
> To find the image of any point (a, b), find $\mathbf{M}\begin{bmatrix} a \\ b \end{bmatrix}$

This table lists some simple matrices and the geometrical transformations they represent.

$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$	Identity: all points unchanged
$\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$	Rotation of 90° anticlockwise about O
$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$	Rotation of 180° about O
$\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$	Rotation of 90° clockwise about O
$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$	Reflection in the x-axis
$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$	Reflection in the y-axis
$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$	Reflection in the line $y = x$
$\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$	An enlargement, ×2, centre at O
$\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}$	A stretch, ×3, from the y-axis
$\begin{bmatrix}1 & 0\\0 & 4\end{bmatrix}$	A stretch, ×4, from the x-axis
$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$	A shear, parallel to the x-axis

Transformations represented by matrices can be combined by multiplying the matrices in a special order. Suppose you want to transform a point $\begin{bmatrix}x\\y\end{bmatrix}$ first by the transformation represented by **A**, and then by the transformation represented by **B.**

$\begin{bmatrix}x\\y\end{bmatrix}$ would become $\mathbf{A}\begin{bmatrix}x\\y\end{bmatrix}$ and then $\mathbf{B}\left(\mathbf{A}\begin{bmatrix}x\\y\end{bmatrix}\right)$. By the associativity of matrix multiplication,

$\mathbf{B}\left(\mathbf{A}\begin{bmatrix}x\\y\end{bmatrix}\right) = (\mathbf{BA})\begin{bmatrix}x\\y\end{bmatrix}$.

The matrix **BA** represents the transformation **A** followed by the transformation **B**

Example 1.4.2

Find the matrix which represents a rotation of 90° anticlockwise about O, followed by a reflection in the x-axis, followed by a shear parallel to the x-axis such that (0, 1) is transformed to (1, 1).

Solution

Successively multiply any point (a, b) by the matrices $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

By associativity, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right) \right\} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix}$.

The single matrix is therefore

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}.$$

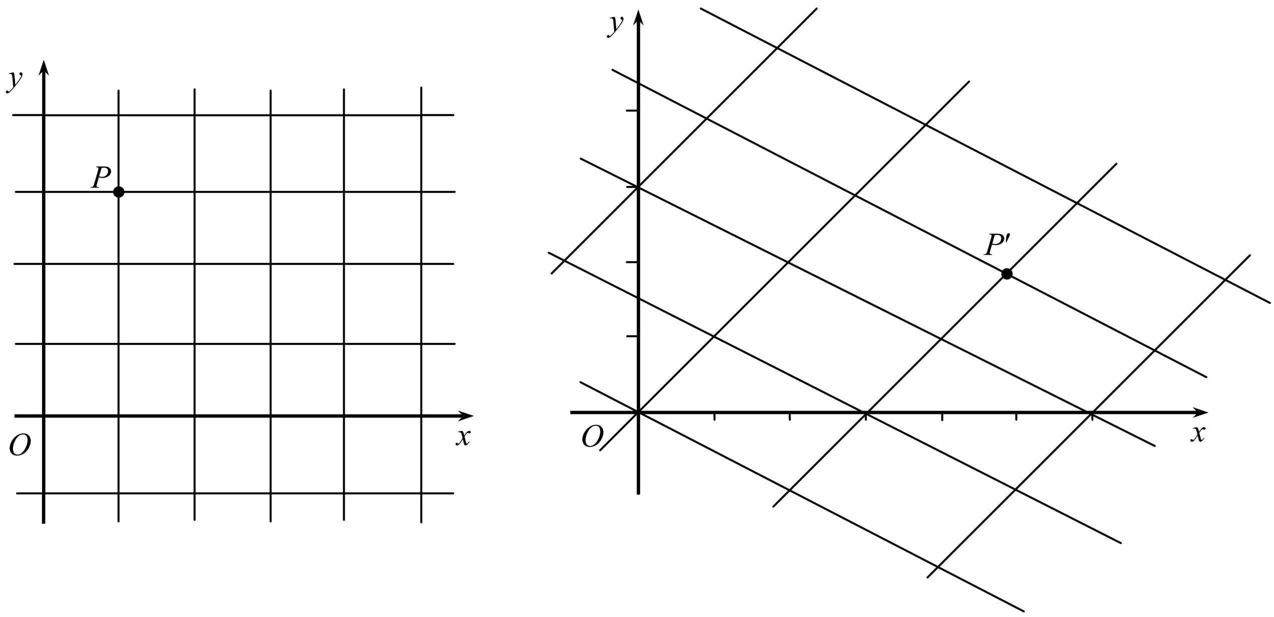
1.5 Linear transformations

A matrix such as $\mathbf{M} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ maps the unit vectors as shown:

$$\mathbf{M} : \mathbf{i} \to \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\mathbf{i} - \mathbf{j},$$

$$\mathbf{M} : \mathbf{j} \to \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{i} + \mathbf{j}.$$

Any point, say P, on the usual Cartesian grid is then mapped onto the corresponding point, P', on a grid of parallelograms defined by $2\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$.



Any transformation which maps the Cartesian grid of straight lines onto another such grid of parallel straight lines is called a **linear** transformation. A more formal definition is given below.

> A linear transformation has the following properties:
>
> For any vectors **a** and **b**, and any scalar λ,
>
> $$\mathrm{T}(\lambda\mathbf{a}) = \lambda\mathrm{T}(\mathbf{a}),$$
> $$\mathrm{T}(\mathbf{a}+\mathbf{b}) = \mathrm{T}(\mathbf{a}) + \mathrm{T}(\mathbf{b})$$

Typical linear transformations are rotations about the origin, reflections in lines through the origin, stretches and shears.

Any linear transformation can be represented by a square matrix. Furthermore, as illustrated by the matrix **M** at the start of this section,

> The matrix representing a given linear transformation in two dimensions is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where the first column is the image of **i** and the second column is the image of **j**

Example 1.5.1

Find the matrix **M** representing a shear parallel to the y-axis such that $(1,\ 0) \to (1,\ 4)$.

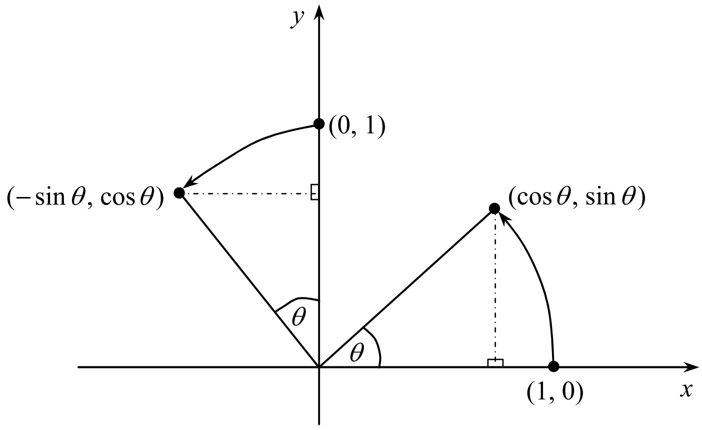
Solution

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ so } \mathbf{M} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}.$$

Example 1.5.2

Find the matrix **R** which represents an anticlockwise rotation of θ about the origin.

Solution

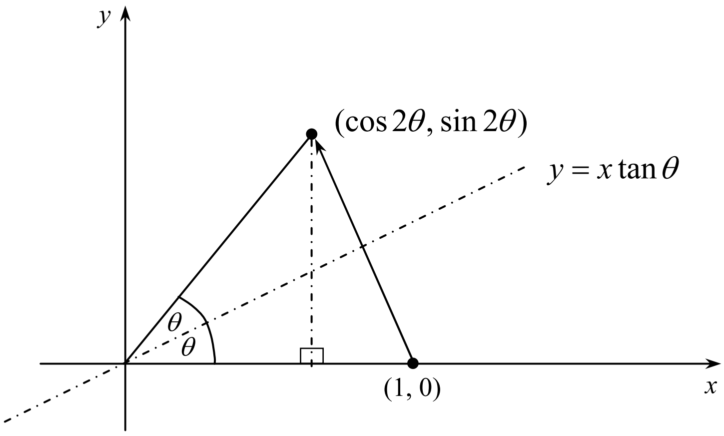


$$\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix} \text{ and } \begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}. \text{ So, } \mathbf{R}=\begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}.$$

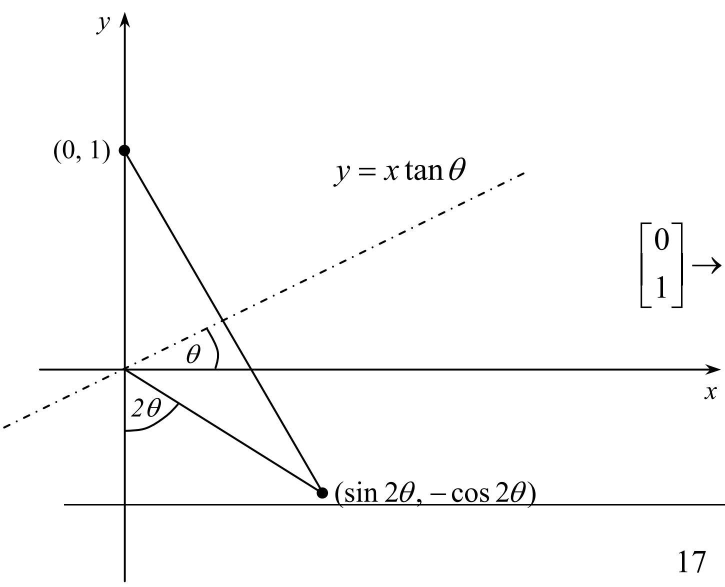
Example 1.5.3

Find the matrix **M** which represents a reflection in the line $y=\tan\theta$.

Solution



$$\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}\cos 2\theta\\\sin 2\theta\end{bmatrix}$$



$$\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}\sin 2\theta\\-\cos 2\theta\end{bmatrix}, \text{ so } \mathbf{M}=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\\sin 2\theta & -\cos 2\theta\end{bmatrix}.$$

You should know the following transformation matrices:

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Rotation of θ anticlockwise about O
$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$	Reflection in the line $y = x\tan\theta$
$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$	Enlargement, $\times\lambda$, centre O
$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	Stretch, $\times k$, from the x-axis
$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	Stretch, $\times k$, from the y-axis
$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	Shear, parallel to the x-axis
$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$	Shear, parallel to the y-axis

Exercise 1D

1. A rotation of 90° anticlockwise about O is represented by $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (a) Find $\mathbf{M}^2$. What transformation is represented by $\mathbf{M}^2$?

   (b) Find $\mathbf{M}^3$. What transformation is represented by $\mathbf{M}^3$?

2. Reflections in the x-axis, the y-axis and in the line $y = x$ are given, respectively, by

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

   Find $\mathbf{L}^2$, $\mathbf{M}^2$ and $\mathbf{N}^2$ and explain your results.

3. Describe the geometrical transformations represented by the matrices
   (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

4. Show that the transformation represented by the matrix $\mathbf{M} = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$, where a and b are constants, transforms all points onto a line. Find the equation of this line.

5. Write down matrices which represent the following transformations:
   (a) reflection in $y = x\sqrt{3}$,
   (b) anticlockwise rotation of 30° about the origin,
   (c) reflection in $y = -x$.

6. Calculate the matrix which represents an anticlockwise rotation of θ° about the origin, followed by a reflection in the x-axis, and a clockwise rotation of θ° about the origin. Hence describe the combined transformation as simply as possible.

7. Show that a reflection in the line $y = x\tan\theta$ followed by a reflection in the line $y = x\tan\phi$ is equivalent to a rotation about the origin. Find the angle of the rotation.

8. A rotation of θ° is represented by the matrix

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

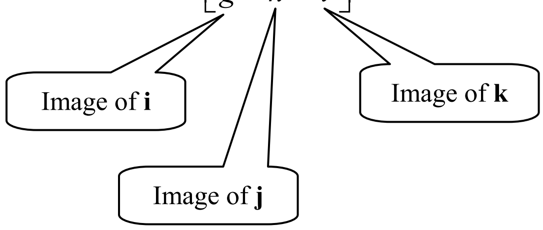
   Use the product of $\mathbf{R}$ with itself to prove that
   (a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$,
   (b) $\sin 2\theta = 2\sin\theta\cos\theta$.

1.6 Matrix transformation (3-dimensional)

The ideas presented in this chapter extend in a natural way to linear transformations of three dimensional space.

The matrix **M** representing a given linear transformation has columns given by the images of **i**, **j**, and **k**.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$



Example 1.6.1

Find the matrix **M** representing a reflection in the plane $z = 0$.

Solution

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Example 1.6.2

Find the matrix **R** representing a rotation of θ° about the x-axis.

Solution

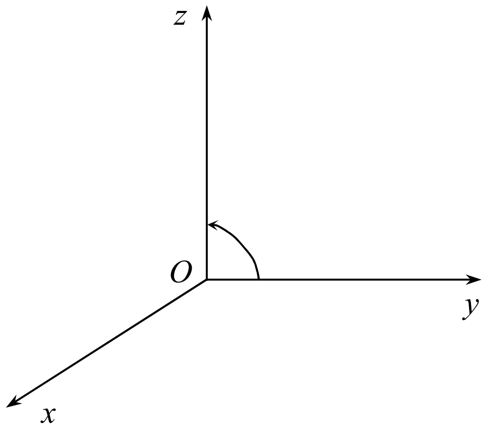
$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

You should know the following transformation matrices:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	Identity
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Rotations of θ° about the x-, y- and z-axes, respectively
$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$	Enlargement, scale factor λ
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	Reflections in the planes $x = 0$, $y = 0$ and $z = 0$ respectively
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	Reflections in the planes $x = y$, $y = z$ and $x = z$ respectively

Miscellaneous exercises 1

1. (a) Write down the 3×3 matrix that represents a rotation of 90° about the x-axis in the direction of y to z as shown in the diagram.



(b) The matrix

$$\begin{bmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

represents a rotation about the y-axis through an acute angle θ. Show that $\theta = \frac{\pi}{3}$.

(c) The transformation in part (a) is denoted by T_1 and the transformation in part (b) is denoted by T_2. Find the matrix which represents the transformation T_1 followed by T_2.

[AQA–NEAB, 2001]

2. The matrix $\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Two matrices, **X** and **Y**, are said to commute if $\mathbf{XY} = \mathbf{YX}$.

The 2×2 matrix $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, commutes with **A**. Show that $b = 2c$ and find a relationship between a, c and d.

[AQA–AEB, 2000]

3. The matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ represents a rotation.

(a) Find the equation of the axis of this rotation.

(b) What is the angle of the rotation?

4. The transformation with matrix $\mathbf{T}$, where $\mathbf{T} = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$, maps the point (x, y) onto the point (x', y'), where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Find the equation of the line onto which the line $y + x = 0$ is mapped by the transformation.

(b) Find the values of m for which the line $y = mx$ is mapped onto itself.

[JMB, 1969]

5. The transformation with matrix $\mathbf{T}$, where $\mathbf{T} = \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$, maps the point (x, y) onto the point (x', y'), where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) Find the equation of the image of the line $y = 3x$ under this transformation.

(b) Find also the equations of the lines through the origin which are turned through a right angle about the origin under this transformation.

[JMB, 1979]

6. The transformation with matrix $\mathbf{T}$ maps the point (x, y) onto the point (x', y'), where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{T} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) By considering the images of the points $A(1, 0)$ and $B(0, 1)$, or otherwise, determine the geometrical transformation represented by each of the matrices

$$\mathbf{T}_1(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

$$\mathbf{T}_2(\phi) = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}.$$

(b) Verify by matrix multiplication that

$$\mathbf{T}_2(p)\mathbf{T}_2(q) = \mathbf{T}_1\left[2(p-q)\right],$$

$$\mathbf{T}_2(p)\mathbf{T}_1(q) = \mathbf{T}_1(-q)\mathbf{T}_2(p),$$

and interpret these results geometrically.

[JMB, 1973]

7. (a) A transformation, T_1, of three dimensional space is given by $\mathbf{r}' = \mathbf{M}\mathbf{r}$, where

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Describe the transformation geometrically.

(b) Two other transformations are defined as follows: T_2 is a reflection in the x–z plane, and T_3 is a rotation through 180° about the line $x = 0$, $y + z = 0$. By considering the image under each transformation of the points with position vectors **i**, **j**, **k**, or otherwise, find the matrix for each of T_2 and T_3.

(c) Determine the matrices for the combined transformations $T_3\ T_1$ and $T_1\ T_3$ and describe each of these transformations geometrically.

[JMB, 1978]

Chapter 2: The Vector Product

This chapter introduces the idea of a product of two vectors which is itself a vector. When you have completed it, you will:

- know what is meant by vector product;
- know that vector product is distributive over addition;
- be able to calculate the vector product in coordinate form;
- be able to find areas using vector products;
- know how to use scalar triple products to find volumes of parallelepipeds.

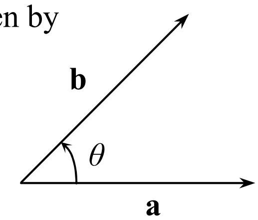
Further work on vector products is given in Chapter 4.

2.1 Introduction

You will have met already one method of 'multiplying' two vectors, called the **scalar product**.

The scalar product of two vectors, **a** and **b**, is given by

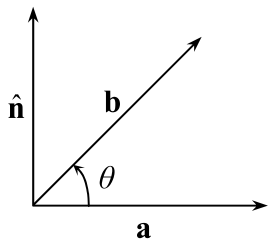
$$\mathbf{a.b} = ab\cos\theta$$



where a and b are the magnitudes of **a** and **b**, respectively, and θ is the angle between them. You should note that $\mathbf{a.b}$ is a scalar quantity, hence the name 'scalar product'.

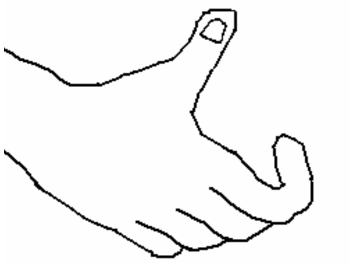
There is also a way of multiplying two vectors to give a vector quantity. This method is called the **vector product** and is defined as follows.

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\hat{\mathbf{n}}$$

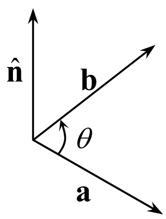


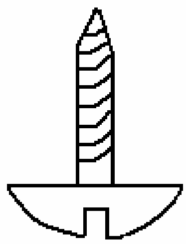
The vector $\hat{\mathbf{n}}$ is a unit vector, perpendicular to **a** and **b** in the sense shown above.

There are, of course, two opposite directions both perpendicular to **a** and **b**. The direction of $\hat{\mathbf{n}}$ is that given by the thumb of a right hand when the fingers are turning from **a** to **b**.



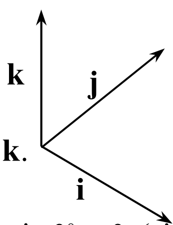
A right hand. The fingers point from **a** to **b**. The thumb gives the direction of $\hat{\mathbf{n}}$.





A normal (right-handed) screw is driven in by turning clockwise, i.e. from **a** to **b**.

The three unit vectors **i**, **j** and **k** form a right-hand set.



So, $\mathbf{i}\times\mathbf{j} = 1\times1\times\sin 90^\circ\,\mathbf{k} = \mathbf{k}$.

Note also, that $\mathbf{i}\times\mathbf{i} = 1\times1\times\sin 0^\circ = 0$ (since $\sin 0^\circ = 0$).

Exercise 2A

1. (a) Find the nine possible products $\mathbf{a} \times \mathbf{b}$, where each of $\mathbf{a}$ and $\mathbf{b}$ are one of the vectors $\mathbf{i}$, $\mathbf{j}$ or $\mathbf{k}$.

   (b) Summarise what you notice about your answers.

2.2 Properties of the vector product

In Exercise 2A, you should have found the following results for vector products of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$$

$$\mathbf{i}\times\mathbf{j} = \mathbf{k},\ \mathbf{j}\times\mathbf{k} = \mathbf{i},\ \mathbf{k}\times\mathbf{i} = \mathbf{j}$$

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k},\ \mathbf{k}\times\mathbf{j} = -\mathbf{i},\ \mathbf{i}\times\mathbf{k} = -\mathbf{j}$$

The first three of these results are special cases of a general result about the vector product of parallel vectors. If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then the angle θ between them is either 0° or 180° and therefore $\sin\theta = 0$. Thus, $\mathbf{a}\times\mathbf{b} = \mathbf{0}$.

$$\mathbf{a}\times\mathbf{b} = 0 \Leftrightarrow \mathbf{a} = 0 \text{ or } \mathbf{b} = 0 \text{ or } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel}$$

The other six results concerning the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ can be remembered from the following diagram.

$$\longrightarrow \quad +\text{ve}$$

$$\ldots\ \mathbf{i} \quad \mathbf{j} \quad \mathbf{k} \quad \mathbf{i} \quad \mathbf{j} \quad \mathbf{k}\ \ldots$$

$$\longleftarrow \quad -\text{ve}$$

If two adjacent vectors are multiplied, then they equal the next vector to the right. For example:

$$\ldots\ \mathbf{i} \quad \mathbf{j} \quad \boxed{\mathbf{k} \quad \mathbf{i} \quad \mathbf{j}} \quad \mathbf{k} \quad \ldots \ \Rightarrow \mathbf{k}\times\mathbf{i} = \mathbf{j}$$

However, multiplying in the reverse order gives minus the next vector. For example:

$$\mathbf{i}\times\mathbf{k} = -\mathbf{j}$$

In general, the direction of $\mathbf{b}\times\mathbf{a}$ is opposite to the direction of $\mathbf{a}\times\mathbf{b}$, and so:

For any two vectors, $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$

A familiar operation in ordinary arithmetic and algebra is that of multiplying out brackets.

For example, $2(x+3)=2x+6$. This property is called the **distributivity** of multiplication over addition.

In fact, the operation of vector product is also distributive over vector addition and so you can perform algebraic steps such as:

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{c}+\mathbf{d})=\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{d}+\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{d}.$$

Note that you **must** keep the order of the symbols the same, i.e. $\mathbf{a}\times\mathbf{c}$ not $\mathbf{c}\times\mathbf{a}$. For much of the rest of this chapter you should assume this property of distributivity. It will be proved in Section 2.7.

Example 2.2.1

Find $(\mathbf{i}+\mathbf{j})\times(\mathbf{j}+\mathbf{k})$.

Solution

$$\mathbf{i}\times\mathbf{j}+\mathbf{i}\times\mathbf{k}+\mathbf{j}\times\mathbf{j}+\mathbf{j}\times\mathbf{k}=\mathbf{k}-\mathbf{j}+\mathbf{0}+\mathbf{i}=\mathbf{i}-\mathbf{j}+\mathbf{k}$$

2.3 Vector products in component form

The distributivity of the vector product over addition can be used to find the value of expressions such as $2\mathbf{i}\times 3\mathbf{j}$.

$$\begin{aligned}2\mathbf{i}\times 3\mathbf{j}&=(\mathbf{i}+\mathbf{i})\times(\mathbf{j}+\mathbf{j}+\mathbf{j})\\&=\mathbf{i}\times\mathbf{j}+\mathbf{i}\times\mathbf{j}+\mathbf{i}\times\mathbf{j}+\mathbf{i}\times\mathbf{j}+\mathbf{i}\times\mathbf{j}+\mathbf{i}\times\mathbf{j}\quad\text{(six terms)}\\&=6\mathbf{i}\times\mathbf{j}.\end{aligned}$$

In general:

> For λ, μ any scalars, $\lambda\mathbf{a}\times\mu\mathbf{b}=\lambda\mu\ \ \mathbf{a}\times\mathbf{b}$

This result enables you to find the vector product of any two vectors in component form.

Example 2.3.1

Find $(2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\times(3\mathbf{i}-\mathbf{j}+4\mathbf{k})$

Solution

$$\begin{aligned}&6\mathbf{i}\times\mathbf{i}-2\mathbf{i}\times\mathbf{j}+8\mathbf{i}\times\mathbf{k}+9\mathbf{j}\times\mathbf{i}-3\mathbf{j}\times\mathbf{j}+12\mathbf{j}\times\mathbf{k}-6\mathbf{k}\times\mathbf{i}+2\mathbf{k}\times\mathbf{j}-8\mathbf{k}\times\mathbf{k}\\&=-2\mathbf{k}-8\mathbf{j}-9\mathbf{k}+12\mathbf{i}-6\mathbf{j}-2\mathbf{i}\\&=10\mathbf{i}-14\mathbf{j}-11\mathbf{k}.\end{aligned}$$

Example 2.3.2

Find $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Solution

$$\begin{aligned} (\mathbf{i}+2\mathbf{j})\times(\mathbf{i}+3\mathbf{k}) &= \cancel{\mathbf{i}\times\mathbf{i}} + 3\mathbf{i}\times\mathbf{k} + 2\mathbf{j}\times\mathbf{i} + 6\mathbf{j}\times\mathbf{k} \\ &= -3\mathbf{j} - 2\mathbf{k} + 6\mathbf{i} \\ &= \begin{bmatrix} 6 \\ -3 \\ -2 \end{bmatrix}. \end{aligned}$$

Exercise 2B

Find the following vector products.

1. $\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$.

2. $(\mathbf{i}+\mathbf{j}+\mathbf{k})\times\mathbf{i}$.

3. $(3\mathbf{i}+\mathbf{j})\times 2\mathbf{k}$.

4. $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{k})$.

5. $(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})\times(5\mathbf{i}+6\mathbf{j}+7\mathbf{k})$.

6. $(3\mathbf{i}-4\mathbf{j}+6\mathbf{k})\times(2\mathbf{i}+\mathbf{j}+3\mathbf{k})$.

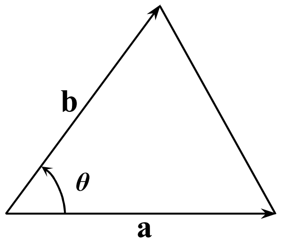
7. $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

8 $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

2.4 Application of vector products to areas

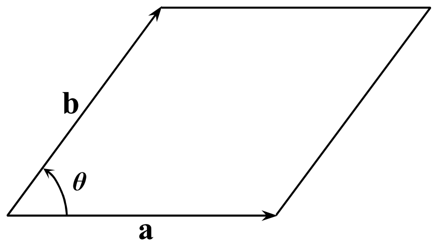
Arguably, the most important use of vector products is in mechanics where moments can conveniently be expressed using these products.

In pure mathematics, you have already met the expression "$ab\sin\theta$" in connection with areas.



Area $= \frac{1}{2}ab\sin\theta = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

> The area of a triangle is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$



Area $= ab\sin\theta = |\mathbf{a}\times\mathbf{b}|$

> The area of a parallelogram is $|\mathbf{a}\times\mathbf{b}|$

To find the areas of triangles or parallelograms using vector products, it is therefore necessary to first find two vectors representing adjacent edges.

Example 2.4.1

Find the area of triangle ABC, where A is (2, 0, 3), B is (1, −3, 4) and C is (−1, 2, 0).

Solution

$$\overrightarrow{AB} = \mathbf{B}-\mathbf{A} = (\mathbf{i}-3\mathbf{j}+4\mathbf{k})-(2\mathbf{i}+3\mathbf{k}) = -\mathbf{i}-3\mathbf{j}+\mathbf{k}$$

and

$$\overrightarrow{AC} = \mathbf{C}-\mathbf{A} = (-\mathbf{i}+2\mathbf{j})-(2\mathbf{i}+3\mathbf{k}) = -3\mathbf{i}+2\mathbf{j}-3\mathbf{k}$$

Then

$$\begin{aligned}\overrightarrow{AB}\times\overrightarrow{AC} &= -2\mathbf{i}\times\mathbf{j}+3\mathbf{i}\times\mathbf{k}+9\mathbf{j}\times\mathbf{i}+9\mathbf{j}\times\mathbf{k}-3\mathbf{k}\times\mathbf{i}+2\mathbf{k}\times\mathbf{j}\\ &= -2\mathbf{k}-3\mathbf{j}-9\mathbf{k}+9\mathbf{i}-3\mathbf{j}-2\mathbf{i}\\ &= 7\mathbf{i}-6\mathbf{j}-11\mathbf{k}.\end{aligned}$$

$$\text{Area of }\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\sqrt{49+36+121} = \frac{1}{2}\sqrt{206}$$

2.5 Triple products

For any two vectors **b** and **c**, $\mathbf{b}\times\mathbf{c}$ is itself a vector. A product can therefore be formed with any third vector, **a** say.

$\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ is a scalar quantity and is therefore called a **scalar triple product**.

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is a vector quantity and is therefore called a **vector triple product**.

To find a triple product, you simply perform each product in turn.

Example 2.5.1

(a) Find $(\mathbf{i}+\mathbf{j}).[(\mathbf{j}+\mathbf{k})\times(\mathbf{i}+\mathbf{k})]$. (b) Find $(\mathbf{i}+\mathbf{j})\times[(\mathbf{j}+\mathbf{k})\times(\mathbf{i}+\mathbf{k})]$.

Solution

(a) $(\mathbf{j}+\mathbf{k})\times(\mathbf{i}+\mathbf{k}) = \mathbf{j}\times\mathbf{i}+\mathbf{j}\times\mathbf{k}+\mathbf{k}\times\mathbf{i}+\mathbf{k}\times\mathbf{k} = -\mathbf{k}+\mathbf{i}+\mathbf{j}$,

$$(\mathbf{i}+\mathbf{j}).(\mathbf{i}+\mathbf{j}-\mathbf{k}) = \begin{bmatrix}1\\1\\0\end{bmatrix}.\begin{bmatrix}1\\1\\-1\end{bmatrix} = (1\times1)+(1\times1)+(0\times-1) = 1+1 = 2.$$

(b) $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{j}-\mathbf{k}) = (\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{j}) - \mathbf{i}\times\mathbf{k} - \mathbf{j}\times\mathbf{k} = -\mathbf{i}+\mathbf{j}$

As you will see in Exercise 2C, $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ are not necessarily the same. It is therefore essential to use brackets in a vector triple product. However, this is not true for scalar triple products.

An expression such as $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ could mean either $(\mathbf{a}.\mathbf{b})\times\mathbf{c}$ or $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$. However, $\mathbf{a}.\mathbf{b}$ is a scalar quantity and so the vector product $(\mathbf{a}.\mathbf{b})\times\mathbf{c}$ is not possible. Brackets are therefore not usually shown in scalar triple products.

$\mathbf{a}.\mathbf{b}\times\mathbf{c}$ means $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$

Exercise 2C

1. Find the area of triangle *ABC*, where *A* is (1, 1, 2), *B* is (5, –1, 3) and *C* is (1, –2, 1).

2. Find the following triple products.
   (a) $(2\mathbf{i}+\mathbf{j}).(\mathbf{j}+\mathbf{k})\times\mathbf{i}$
   (b) $(\mathbf{i}-\mathbf{j}).(\mathbf{i}+2\mathbf{j})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$
   (c) $(\mathbf{i}+3\mathbf{j}+\mathbf{k}).(2\mathbf{i}-\mathbf{j})\times(\mathbf{i}-4\mathbf{j}-\mathbf{k})$

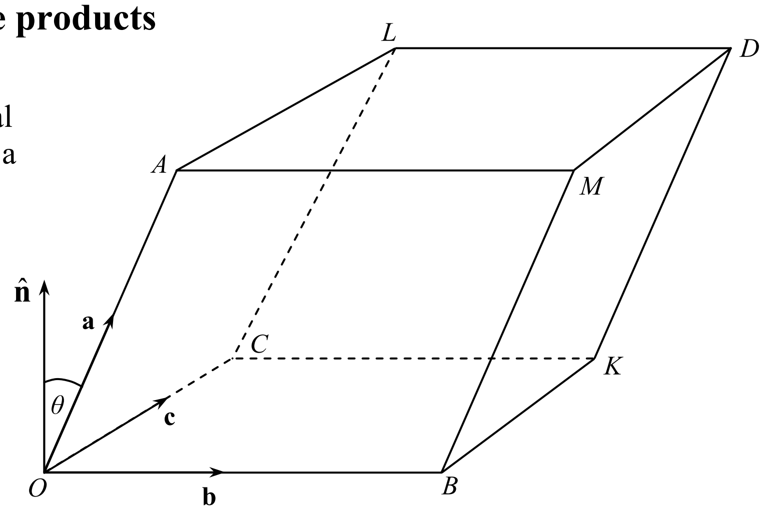
3. Find vectors **a**, **b** and **c** such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

4. (a) Find $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ and $\mathbf{a}\times\mathbf{b}.\mathbf{c}$ for each of the following sets of vectors.
   (i) $\mathbf{a}=\mathbf{i}+\mathbf{j}$, $\mathbf{b}=\mathbf{j}+\mathbf{k}$, $\mathbf{c}=\mathbf{i}+\mathbf{k}$.
   (ii) $\mathbf{a}=2\mathbf{i}-\mathbf{j}$, $\mathbf{b}=\mathbf{j}+\mathbf{k}$, $\mathbf{c}=\mathbf{j}-\mathbf{k}$.
   (iii) $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=\mathbf{i}$, $\mathbf{c}=2\mathbf{j}+\mathbf{k}$.

   (b) What can you conclude from your answers to part (a).

2.6 Properties of scalar triple products

A parallelepiped is a three dimensional shape with six faces, each of which is a parallelogram



Consider the parallelepiped shown with adjacent edges $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. Let $\mathbf{b}\times\mathbf{c} = |\mathbf{b}\times\mathbf{c}|\ \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to plane *OBKC*, and suppose that the angle θ between **a** and $\hat{\mathbf{n}}$ is acute. Then,

$$\begin{aligned}\mathbf{a.b}\times\mathbf{c} &= \mathbf{a.}|\mathbf{b}\times\mathbf{c}|\hat{\mathbf{n}}\\ &= |\mathbf{b}\times\mathbf{c}|\mathbf{a.}\hat{\mathbf{n}}\\ &= |\mathbf{b}\times\mathbf{c}||\mathbf{a}|\cos\theta.\end{aligned}$$

The volume of a parallelepiped is given by

area of base × perpendicular height.

The volume of the parallelepiped shown above is therefore given by $\mathbf{a.b}\times\mathbf{c}$. Similarly, it is also given by $\mathbf{b.c}\times\mathbf{a}$ and $\mathbf{c.a}\times\mathbf{b}$. All three of these scalar triple products must therefore be equal.

$$\mathbf{a.b}\times\mathbf{c} = \mathbf{b.c}\times\mathbf{a} = \mathbf{c.a}\times\mathbf{b}$$

You should note that these equal triple products all involve the same cyclic order of the three vectors, i.e. … **a b c a b c** …

If this order is changed, then the sign of the scalar triple product is changed. For example,

$$\mathbf{a.c}\times\mathbf{b} = \mathbf{a.}-(\mathbf{b}\times\mathbf{c}) = -\mathbf{a.b}\times\mathbf{c}.$$

The volume of a parallelepiped is always positive and so a general formula is given by the modulus of a scalar triple product.

If three adjacent edges of a parallelepiped are represented by the vectors **a**, **b** and **c**, then the volume of the parallelepiped is given by $|\mathbf{a.b}\times\mathbf{c}|$

Example 2.6.1

Show that the dot and cross in a scalar triple product can be interchanged, i.e. $\mathbf{a.b\times c = a\times b.c}$.

Solution

$$\begin{aligned}\mathbf{a\times b.c} &= \mathbf{c.a\times b} && (\textbf{.}\text{ is commutative})\\ &= \mathbf{a.b\times c} && (\text{cyclic interchange})\end{aligned}$$

$$\boxed{\mathbf{a.b\times c = a\times b.c}}$$

2.7 Proof of the distributive law

The scalar product is distributive over addition, i.e. you can expand brackets:

$$\mathbf{a.(b+c) = a.b + a.c}.$$

It is an important property of the vector product that it, also, is distributive over addition. The purpose of this section is to prove this result.

The proof requires use of the distributivity of the scalar product and also the use of scalar triple products.

Example 2.7.1

Prove that $\mathbf{a\times(b+c) = a\times b + a\times c}$.

Solution

Let $\mathbf{r}$ be any vector, then

$$\begin{aligned}\mathbf{r.a\times(b+c)} &= \mathbf{r\times a.(b+c)} && (\text{interchange } \textbf{.} \text{ and } \times)\\ &= \mathbf{r\times a.b + r\times a.c} && (\text{distributivity of } \textbf{.})\\ &= \mathbf{r.a\times b + r.a\times c} && (\text{interchange } \textbf{.} \text{ and } \times)\end{aligned}$$

Hence, $\mathbf{r.(a\times(b+c) - a\times b - a\times c)}$ is zero for **any** vector $\mathbf{r}$.

The vector $\mathbf{a\times(b+c) - a\times b - a\times c}$ is therefore perpendicular to all other vectors and so must be zero, as required.

The vector product is distributive over addition, i.e. $\mathbf{a\times(b+c) = a\times b + a\times c}$ $\mathbf{(a+b)\times c = a\times c + b\times c}$

Exercise 2D

1. Use a similar method to that of Example 2.7.1 to prove that $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$.

Miscellaneous exercises 2

1. The points *A*, *B* and *C* have position vectors $\mathbf{a}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $\mathbf{b}=3\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ and $\mathbf{c}=-\mathbf{i}+4\mathbf{j}-4\mathbf{k}$, respectively.
   (a) Write down the vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ and hence determine
      (i) $(\mathbf{b}-\mathbf{a}).(\mathbf{c}-\mathbf{a})$,
      (ii) $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$.

   (b) Using the results from part (a), or otherwise, find
      (i) the cosine of the acute angle between the line *AB* and the line *AC*, giving your answer in an exact form,
      (ii) the area of triangle *ABC*, giving your answer in an exact surd form,
      (iii) a vector equation of the plane through *A*, *B* and *C*, giving your answer in the form $\mathbf{r}.\mathbf{n}=d$.

[AEB, 1998]

2. (a) Simplify $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})$.
   (b) Given that **a** and **b** are non-zero vectors and that

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=\mathbf{0},$$

   write down the possible values of the angle between **a** and **b**.

[NEAB]

3. Given that

$$\mathbf{a}\times\mathbf{b}=\mathbf{i},\ \mathbf{b}\times\mathbf{c}=\mathbf{j}\ \text{ and }\ \mathbf{c}\times\mathbf{a}=\mathbf{k},$$

   express

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+2\mathbf{b}+3\mathbf{c})$$

   in terms of **i**, **j** and **k**.

[NEAB]

Further questions on this topic can be found in Chapter 4.

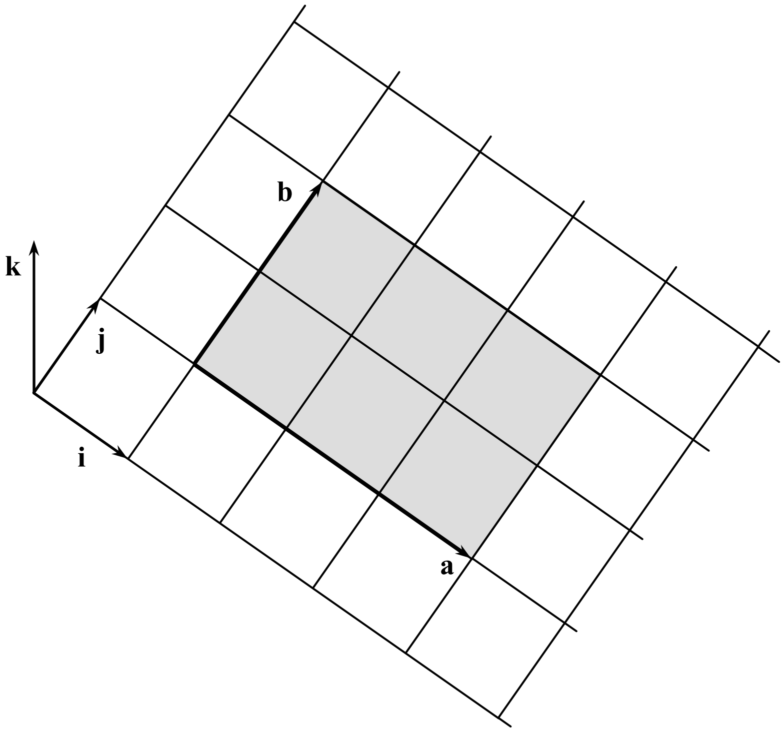
Chapter 3: Determinants

This chapter introduces the idea of the determinant of a matrix. When you have completed it, you will:

- know that the determinant of a 2×2 matrix is the area scale factor;
- know that the determinant of a 3×3 matrix is the volume scale factor;
- be able to calculate determinants of 2×2 and 3×3 matrices;
- know how to use simple rules for manipulating determinants;
- know the connection between 3×3 determinants and the scalar triple product;
- know that the determinant of a product is the product of the determinants.

3.1 Directed areas

The diagram shows a rectangle in the **i**–**j** plane.



Its area, which is obviously 6 square units, can be obtained by using vector products as

$$|\mathbf{a}\times\mathbf{b}| = |3\mathbf{i}\times 2\mathbf{j}| = |6\mathbf{k}| = 6.$$

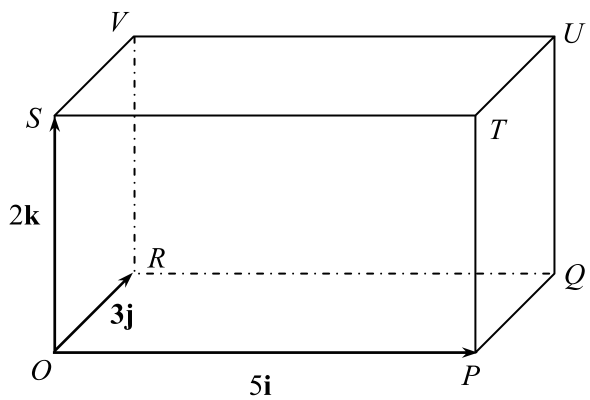
The vector quantity 6**k** tells you more about the rectangle than the scalar quantity 6. It also tells you that it faces in the **k** direction. With this convention, the area of the 'underneath' of the above rectangle is therefore $-6\mathbf{k}$. Note that $\mathbf{a}\times\mathbf{b}$ gives one side of the rectangle, and $\mathbf{b}\times\mathbf{a}$ gives the other side, as determined by the 'right-hand' rule.

Exercise 3A

1. The diagram shows a $2\times3\times5$ cuboid.

   (a) Write down the directed areas of
   (i) *PQUT*, (ii) *ORVS*, (iii) *RQUV*,
   (iv) *OPQR*.

   (b) Express each of the areas in part (a) as a vector product, and hence check your anwers to (a).



3.2 2×2 determinants

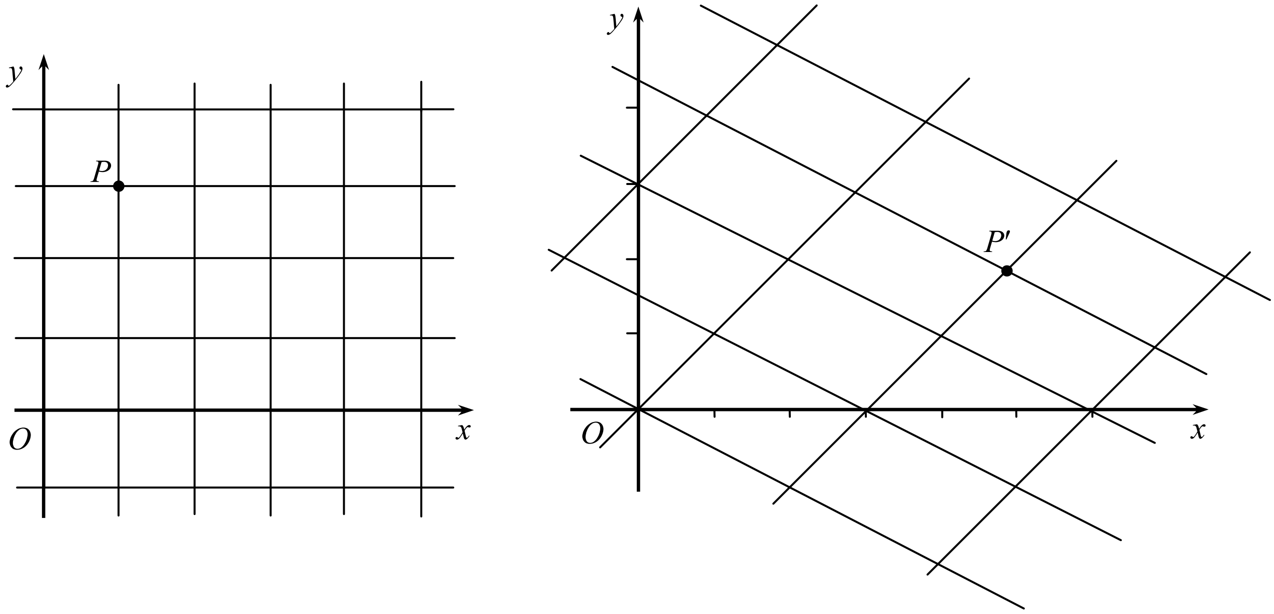
As you have already seen, a 2×2 matrix such as

$$\mathbf{M} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

represents a linear transformation. The transformation represented by $\mathbf{M}$ maps the unit vectors as shown:

$$\mathbf{i} \to 2\mathbf{i} - \mathbf{j},\ \ \mathbf{j} \to \mathbf{i} + \mathbf{j}.$$

Furthermore, any point (say P) on the usual Cartesian grid is mapped onto the corresponding point (P') on a grid of parallelograms defined by $2\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + \mathbf{j}$.



The area scale factor of this transformation is called the **determinant of M** and is denoted by either det(**M**) or $|\mathbf{M}|$. It can be found by comparing an initial directed area (e.g. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$) with the transformed area.

$$(2\mathbf{i} - \mathbf{j}) \times (\mathbf{i} + \mathbf{j}) = 2\mathbf{i} \times \mathbf{j} - \mathbf{j} \times \mathbf{i} = 3\mathbf{k}$$

$$\Rightarrow |\mathbf{M}| = 3 \ \text{ or } \ \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 3.$$

Example 3.2.1

M is the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(a) Find $|\mathbf{M}|$.

(b) Comment on the significance of the sign of your answer to part (a).

Solution

(a) $\mathbf{i} \to \mathbf{j}$ and $\mathbf{j} \to \mathbf{i}$. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ is therefore transformed to $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. $|\mathbf{M}| = -1$.

(b) **M** represents a reflection in the line $y = x$ and so the directions of any areas are reversed. Hence $|\mathbf{M}|$ is negative.

> $|\mathbf{M}|$ is negative if the transformation represented by **M** is either a reflection or involves a reflection.

It is easy to find a general formula for the determinant of a 2×2 matrix. Consider

$$\mathbf{M} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

This maps the unit vectors as shown:

$$\mathbf{i} \to a_1\mathbf{i} + a_2\mathbf{j}, \ \mathbf{j} \to b_1\mathbf{i} + b_2\mathbf{j}$$

Then, $(a_1\mathbf{i} + a_2\mathbf{j}) \times (b_1\mathbf{i} + b_2\mathbf{j}) = a_1b_2\mathbf{i} \times \mathbf{j} + a_2b_1\mathbf{j} \times \mathbf{i}$

$$= (a_1b_2 - a_2b_1)\mathbf{k}.$$

Exercise 3B

1. Find the determinants of the following matrices.

   (a) $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix}$.

2. Which of the transformations represented by the matrices in Question 1 involve a reflection?

3. A matrix **M** represents a rotation of 90º clockwise about the origin, followed by an enlargement of scale factor 2, and then a reflection in the *x*-axis.

   (a) Write down $|\mathbf{M}|$.

   (b) Check your answer to part (a) by finding **M**.

4. $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$.

   (a) Find $|\mathbf{A}|$ and $|\mathbf{B}|$.

   (b) Find $|\mathbf{AB}|$ and $|\mathbf{BA}|$. What do you notice?

3.3 3×3 determinants

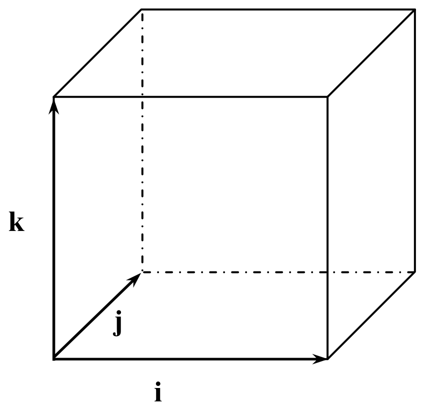
The ideas of the last section can be extended to three dimensions. Let **a**, **b** and **c** be any three vectors:

$$\mathbf{a}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix},\ \mathbf{c}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

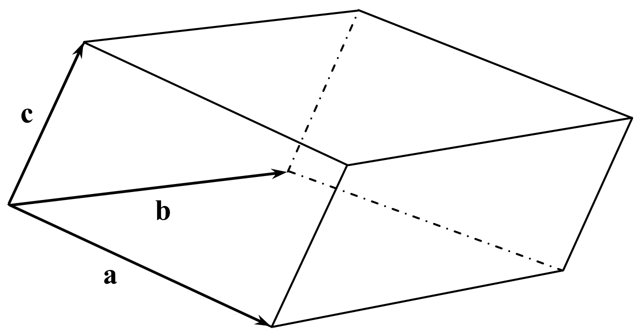
As you have already seen, the 3×3 matrix $\mathbf{M}=\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}$ represents a linear transformation which transforms the unit vectors

$$\mathbf{i}\to\mathbf{a},\ \mathbf{j}\to\mathbf{b},\ \mathbf{k}\to\mathbf{c}.$$

Geometrically, the unit cube is transformed by **M** into a parallelepiped.



volume = 1



volume = $|\mathbf{a}.\mathbf{b}\times\mathbf{c}|$

> The determinant of **M** is defined to be $\mathbf{a}.\mathbf{b}\times\mathbf{c}$ and is written $\det(\mathbf{M})$ or $|\mathbf{M}|$

The determinant of **M** is thus the **volume scale factor** together with a sign (positive or negative). A negative determinant indicates that a reflection is involved in the transformation represented by **M**.

In very simple cases, the determinant can easily be written down once you have pictured the image of the unit cube. For example, consider

$$\mathbf{M}=\begin{bmatrix}2&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

This matrix transforms the unit cube into a cuboid. The volume of the unit cube has been doubled and so $\det(\mathbf{M})=2$.

3.4 A general formula

The determinant of $\mathbf{M}=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is $\mathbf{a}.\mathbf{b}\times\mathbf{c}$.

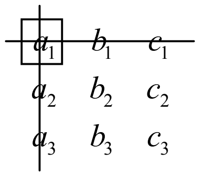
The distributivity of the vector product over addition can be used to find a general formula for $\mathbf{b}\times\mathbf{c}$.

$$\begin{aligned}
\mathbf{b}\times\mathbf{c} &= (b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\times(c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}) \\
&= \cancel{b_1c_1\mathbf{i}\times\mathbf{i}}+b_1c_2\mathbf{i}\times\mathbf{j}+b_1c_3\mathbf{i}\times\mathbf{k} \\
&\quad +b_2c_1\mathbf{j}\times\mathbf{i}+\cancel{b_2c_2\mathbf{j}\times\mathbf{j}}+b_2c_3\mathbf{j}\times\mathbf{k} \\
&\quad +b_3c_1\mathbf{k}\times\mathbf{i}+b_3c_2\mathbf{k}\times\mathbf{j}+\cancel{b_3c_3\mathbf{k}\times\mathbf{k}} \\
&= (b_2c_3-b_3c_2)\mathbf{i}-(b_1c_3-b_3c_1)\mathbf{j}+(b_1c_2-b_2c_1)\mathbf{k} \\
&= \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\mathbf{i}-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}\mathbf{j}+\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\mathbf{k}.
\end{aligned}$$

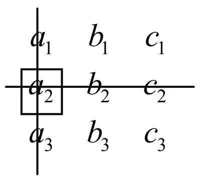
So $|\mathbf{M}|=\mathbf{a}.\mathbf{b}\times\mathbf{c}$.

$$|\mathbf{M}|=a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Fortunately, there is an easy way to remember this formula. Each of these terms is the product of an element of the matrix with the determinant of the 2×2 matrix formed by deleting a row and a column:

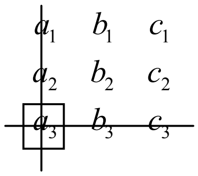


$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$



$$a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

the negative of this term is used



$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

The 'a' column was used to multiply out the determinant. The same process can be used with **any** row or column, the sign of the terms being determined by the pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Example 3.4.1

Multiply out $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -1 \\ 3 & 2 & 0 \end{vmatrix}$.

Solution

Using the second row,

$$-1\begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 6 + (4 \times -9) + (1 \times 7) = -23.$$

In practice, it is usually a good idea to multiply out using a row or column containing as many zeros as possible.

Exercise 3C

1. Write down the determinants of these matrices.

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

2. Repeat Example 3.4.1 using the third column to multiply out the determinant.

3. Find the determinants of

   (a) $\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & 5 \\ 2 & -1 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & -1 \\ 4 & 5 & 3 \end{bmatrix}$.

   (c) What proposition can you make concerning the determinant of the transpose of a matrix?

4. (a) Given that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 4 \\ 2 & 1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, find $|\mathbf{A}|$, $|\mathbf{B}|$ and $|\mathbf{AB}|$.

   (b) What proposition can you make concerning the determinant of the product of two matrices?

3.5 Rules for manipulating determinants

The previous section gives a general procedure for finding any 3×3 determinant. However, it can sometimes be more efficient to first spot ways of simplifying the eventual calculation.

You do not need to know how to prove these results about determinants, although a brief justification is given here for some of these results. The first result was noticed in Exercise 3C.

$$|\mathbf{M}| = |\mathbf{M}^{\mathrm{T}}|$$

A consequence of this rule is that anything you can prove for columns of a determinant must also be true for rows.

> Adding or subtracting any multiple of a row (or column) to another row (or column) does not affect the determinant

For example, consider the addition of $(4 \times \text{column } 2)$ to column 3 for the determinant of

$$\mathbf{M} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

The new determinant is then

$$\begin{aligned} \mathbf{a.b} \times (\mathbf{c} + 4\mathbf{b}) &= \mathbf{a.b} \times \mathbf{c} + \mathbf{a.b} \times 4\mathbf{b} && \text{(distributivity)} \\ &= \mathbf{a.b} \times \mathbf{c} && (\mathbf{b} \times \mathbf{b} = 0) \\ &= |\mathbf{M}| && \text{(as required)} \end{aligned}$$

> Interchanging two rows (or columns) of a matrix changes the sign of the determinant

For example, suppose the first and third columns of the matrix

$$\mathbf{M} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

are interchanged. Then the new determinant is

$$\begin{aligned} \mathbf{c.b} \times \mathbf{a} &= -\mathbf{c.a} \times \mathbf{b} && \text{(property of multiplication)} \\ &= -\mathbf{a.b} \times \mathbf{c} && \text{(cyclic interchange)} \\ &= -|\mathbf{M}| && \text{(as required).} \end{aligned}$$

> Multiplying a row (or column) of a matrix by a scalar multiplies the determinant by that scalar

For example, suppose the second column of the matrix

$$\mathbf{M} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

is doubled. The new determinant is

$$\begin{aligned} \mathbf{a}.2\mathbf{b}\times\mathbf{c} &= \mathbf{a}.2(\mathbf{b}\times\mathbf{c}) \\ &= 2\mathbf{a}.\mathbf{b}\times\mathbf{c} \\ &= 2|\mathbf{M}| \qquad \text{(as required).} \end{aligned}$$

Example 3.5.1

Multiply out (a) $\begin{vmatrix} -1 & 2 & -3 \\ 3 & -1 & 7 \\ -1 & 2 & -3 \end{vmatrix}$, (b) $\begin{vmatrix} 4 & 8 & 16 \\ -1 & -2 & 6 \\ 3 & 1 & 4 \end{vmatrix}$.

Solution

(a) Subtract row 3 from row 1: $\begin{vmatrix} 0 & 0 & 0 \\ 3 & -1 & 7 \\ -1 & 2 & -3 \end{vmatrix} = 0.$

(b) Factorise row 1: $4\begin{vmatrix} 1 & 2 & 4 \\ -1 & -2 & 6 \\ 3 & 1 & 4 \end{vmatrix}$,

add row 1 to row 2: $4\begin{vmatrix} 1 & 2 & 4 \\ 0 & 0 & 10 \\ 3 & 1 & 4 \end{vmatrix}$,

using the second row:

$$= 4\times\{-0[(2\times4)-(4\times1)]+0[(1\times4)-(4\times3)]-10[(1\times1)-(2\times3)]\}$$

$$= 4\times[-10]\times[1-6] = 200.$$

This idea can be used to help factorise an algebraic expression given in the form of a determinant.

To factorise a determinant, use row (or column) operations to obtain a row (or column) of elements with a common factor.

Example 3.5.2

Factorise $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Solution

Add rows 2 and 3 to row 1: $\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$,

take out the factor of $a+b+c$: $(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$,

multiply out: $(a+b+c)(ab+ac+bc-a^2-b^2-c^2)$.

Exercise 3D

1. Evaluate (a) $\begin{vmatrix} 5 & 7 & 9 \\ 4 & 5 & 7 \\ 4 & 6 & 8 \end{vmatrix}$, (b) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix}$.

2. Factorise $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$.

3. (a) Find the relationship between row 1 of $\mathbf{M} = \begin{bmatrix} 5 & -6 & 4 \\ 2 & -1 & -3 \\ 3 & -5 & 7 \end{bmatrix}$ and the other two rows.

   (b) Hence explain why $|\mathbf{M}| = 0$.

3.6 Determinants of products

In Exercise 3C, the result $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$ was obtained for two particular matrices **A** and **B**. This is a very important result which is true for both 2×2 and 3×3 matrices. Taking the 3×3 case: consider **A** and **B** to represent transformations, then the volume scale factors of these transformations are $|\mathbf{A}|$ and $|\mathbf{B}|$, respectively. The combined transformation represented by **AB** transforms volumes by the factor $|\mathbf{B}|$ followed by the factor $|\mathbf{A}|$, i.e. by $|\mathbf{A}||\mathbf{B}|$. Hence:

> The determinant of a product is the product of the determinants,
> $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$

You have seen that, for matrix multiplication, **AB** and **BA** are not necessarily the same. However,

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

$$\text{and } |\mathbf{BA}| = |\mathbf{B}||\mathbf{A}|.$$

Therefore $|\mathbf{AB}|$ and $|\mathbf{BA}|$ are the same, and the matrices **AB** and **BA** do have at least one property in common – their determinant.

Example 3.6.1

Explain why it is impossible to find matrices **X**, **Y** and **Z** to solve the equations

(a) $\mathbf{XY} = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ and $\mathbf{YX} = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}\mathbf{Z} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

Solution

(a) $\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 11$, whereas $\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} = -11$.

(b) $\begin{vmatrix} 3 & 6 \\ 5 & 10 \end{vmatrix} = 0$ and so $\left|\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}\mathbf{Z}\right| = 0$, whereas $\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$.

Miscellaneous exercises 3

1. Find the values of each of these determinants.

(a) $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ -3 & 1 & 4 \end{vmatrix}$, (b) $\begin{vmatrix} p & 2p \\ 3p & 4p \end{vmatrix}$, (c) $\begin{vmatrix} p & 2q & 3r \\ 2p & 3q & 4r \\ 3p & 4q & 6r \end{vmatrix}$.

2. Express the following determinant as the product of four linear factors:

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}.$$

3. (a) Express the determinant $D = \begin{vmatrix} a^3+a^2 & a & 1 \\ b^3+b^2 & b & 1 \\ c^3+c^2 & c & 1 \end{vmatrix}$ as the product of four linear factors.

(b) Given that no two of a, b and c are equal and that $D = 0$, find the value of $a+b+c$.

[AQA]

4. Show that $\begin{vmatrix} k+4 & 5k+7 & k+1 \\ k+2 & 4k+7 & k \\ k+1 & 4k+5 & k-1 \end{vmatrix}$ has the same value for all values of k.

5. Factorise each of these determinants.

(a) $\begin{vmatrix} a & b & c \\ b+c & a+c & a+b \\ bc & ac & ab \end{vmatrix}$, (b) $\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$.

6. The numbers a, b and c are all different and $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = 0$.

Show that $ab + bc + ca = 0$.

7. Solve the equation $\begin{vmatrix} 0 & x-1 & x^2-1 \\ 2x & x & (x+1)^2 \\ 1-x & 1 & 0 \end{vmatrix} = 0$.

Chapter 4: Applications of Vectors

This chapter extends the ideas of Chapter 2. When you have completed it, you will:

- be able to calculate vector products quickly and accurately;
- understand that the scalar triple product is a determinant;
- be able to find the volumes of solids of various shapes;
- know how to determine if three vectors are coplanar;
- be able to find the perpendicular to a plane;
- know how to find the equation of a line in direction ratio form;
- know how to find the line of intersection of two planes;
- be able to find the shortest distance between lines.

4.1 Calculation of vector products in practice

The distributivity of the vector product over addition can be used to find the vector product of any two vectors, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$.

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)\times\left(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}\right) \\
&= \cancel{a_1b_1\mathbf{i}\times\mathbf{i}} + a_1b_2\mathbf{i}\times\mathbf{j} + a_1b_3\mathbf{i}\times\mathbf{k} \\
&\quad + a_2b_1\mathbf{j}\times\mathbf{i} + \cancel{a_2b_2\mathbf{j}\times\mathbf{j}} + a_2b_3\mathbf{j}\times\mathbf{k} \\
&\quad + a_3b_1\mathbf{k}\times\mathbf{i} + a_3b_2\mathbf{k}\times\mathbf{j} + \cancel{a_3b_3\mathbf{k}\times\mathbf{k}} \\
&= \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}
\end{aligned}$$

N.B. $\mathbf{k}\times\mathbf{k} = \mathbf{0}$

This formula probably looks difficult to remember. However, it can be thought of as

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

This form is the one usually used to calculate vector products.

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

This formula is directly related to a determinant formula for the scalar triple product:

$$\mathbf{a}.\mathbf{b}\times\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 4.1.1

Find (a) $\left(3\mathbf{i} + \mathbf{j}\right)\times\left(2\mathbf{i} - \mathbf{j} - \mathbf{k}\right)$ (b) $\left(2\mathbf{i} + \mathbf{j}\right).\left(4\mathbf{i} + \mathbf{k}\right)\times\left(5\mathbf{i} - \mathbf{j} + 2\mathbf{k}\right)$.

Solution

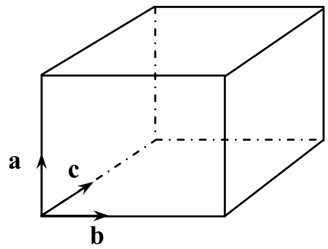
(a) $$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 2 & -1 & -1 \end{vmatrix} = -\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}.$$

(b) $$\begin{vmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 5 & -1 & 2 \end{vmatrix} = 2(0+1) - 1(8-5) = -1.$$

4.2 Volumes

The volumes of a number of solids can be calculated using scalar triple products – i.e. determinants.

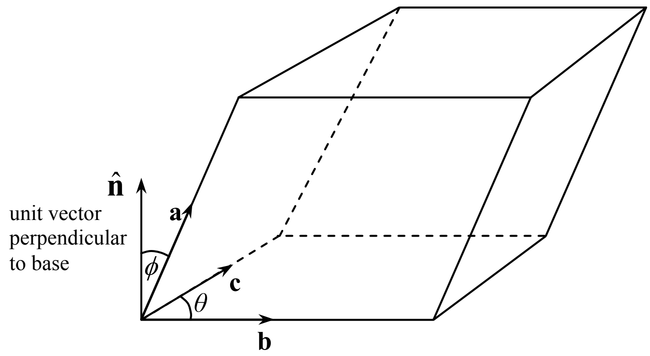
- **Cuboid**



Volume = area of base × height

- area of base = $bc = |\mathbf{b}\times\mathbf{c}|$
- height = a
- volume = abc = $|\mathbf{a}.\mathbf{b}\times\mathbf{c}|$.

- **Parallelepiped**



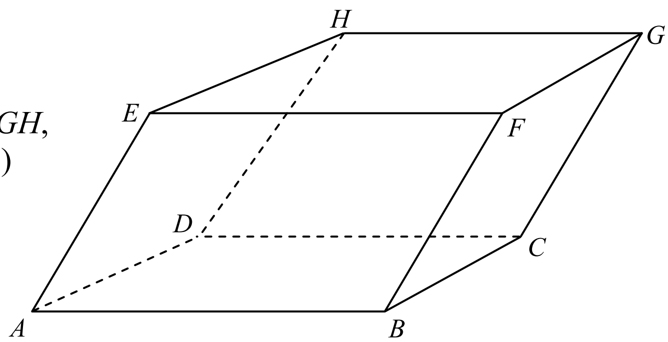
Volume = area of base × perpendicular height

- area of base $= bc\sin\theta = |\mathbf{b}\times\mathbf{c}|$
- perpendicular height $= a\cos\phi = |\mathbf{a}.\hat{\mathbf{n}}|$
- then $\mathbf{b}\times\mathbf{c} = bc\sin\theta\,\hat{\mathbf{n}}$
- volume $= |\mathbf{a}.\mathbf{b}\times\mathbf{c}|$, is the modulus of

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Example 4.2.1

Find the volume of the parallelepiped *ABCDEFGH*, where *A* is (1, –1, 4), *B* is (2, 0, 7), *D* is (5, 0, –4) and *E* is (6, 1, 8)



Solution

First find the three vectors along three sides meeting at one point.

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \overrightarrow{AD} = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}, \quad \overrightarrow{AE} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}.$$

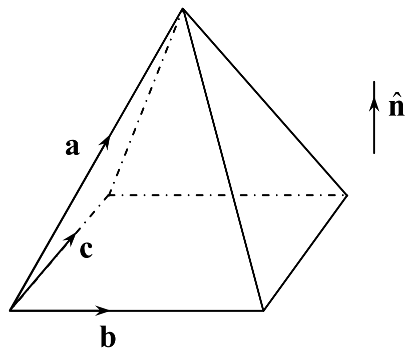
Then

$$\begin{vmatrix} 1 & 1 & 3 \\ 4 & 1 & -8 \\ 5 & 2 & 4 \end{vmatrix} = 1(4+16) - 1(16+40) + 3(8-5)$$

$$= -27.$$

Volume $= |-27| = 27$ cubic units.

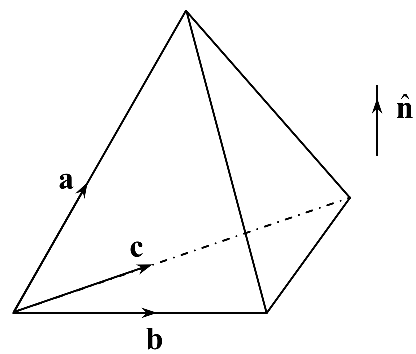
- **Pyramid**



Volume = $\frac{1}{3} \times$ area of base × perpendicular height
(the base is a rectangle or parallelogram)

- volume $= \frac{1}{3}|\mathbf{a}.\mathbf{b} \times \mathbf{c}|$

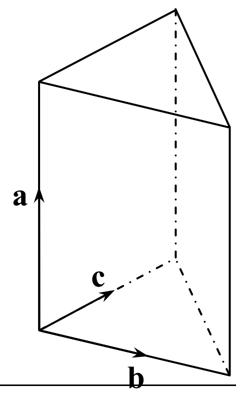
- **Tetrahedron**



Volume = $\frac{1}{3} \times$ area of base × perpendicular height
(the base is a triangle)

- volume $= \frac{1}{6}|\mathbf{a}.\mathbf{b} \times \mathbf{c}|$

- **Triangular prism**

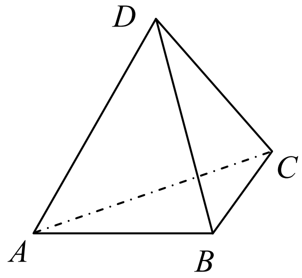


Volume = area of base × height

- area of base $= \frac{1}{2}|\mathbf{a}.\mathbf{b} \times \mathbf{c}|$

Example 4.2.2

Find the volume of the tetrahedron *ABCD* where *A* is (1, 0, 3), *B* is (3, 8, 2), *C* is (4, 1, 1) and *D* is (6, 2, 9).



Solution

$$\text{Volume} = \frac{1}{6}\left|\overrightarrow{AB}\cdot\overrightarrow{AC}\times\overrightarrow{AD}\right|$$

$$= \text{modulus of } \frac{1}{6}\begin{vmatrix} 2 & 8 & -1 \\ 3 & 1 & -2 \\ 5 & 2 & 6 \end{vmatrix}$$

$$= \frac{1}{6}\left|-205\right|$$

$$= \frac{205}{6}.$$

4.3 Coplanarity of vectors

Consider three vectors **a**, **b** and **c**. Form a parallelepiped in which **a**, **b** and **c** are three edges from one point. Then the volume of the parallelepiped is $|\mathbf{a}.\mathbf{b}\times\mathbf{c}|$. This volume is zero if, and only if, **a**, **b** and **c** are coplanar.

$$\mathbf{a},\ \mathbf{b} \text{ and } \mathbf{c} \text{ are coplanar} \Leftrightarrow |\mathbf{a}.\mathbf{b}\times\mathbf{c}| = 0$$

Example 4.3.1

Determine whether or not the following sets of vectors are coplanar.

(a) $2\mathbf{i}-3\mathbf{j}+7\mathbf{k}$, $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $7\mathbf{j}-17\mathbf{k}$

(b) $3\mathbf{i}-2\mathbf{j}+7\mathbf{k}$, $2\mathbf{i}-3\mathbf{j}+4\mathbf{k}$, $5\mathbf{j}+4\mathbf{k}$

Solution

(a)
$$\begin{vmatrix} 2 & -3 & 7 \\ 3 & -1 & 2 \\ 0 & 7 & -17 \end{vmatrix} = -7(4-21)-17(-2+9)$$
$$= 0$$

The vectors are coplanar.

(b)
$$\begin{vmatrix} 3 & -2 & 7 \\ 2 & -3 & 4 \\ 0 & 5 & 4 \end{vmatrix} = -5(12-14)+4(-9+4)$$
$$= -10$$
$$\neq 0$$

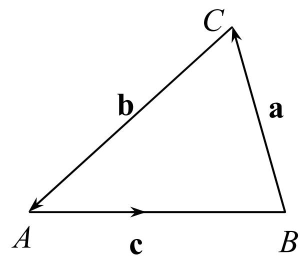
The vectors are not coplanar.

Exercise 4A

1. Calculate (a) $\begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, (b) $(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\times(2\mathbf{i}+4\mathbf{j}-5\mathbf{k})$.

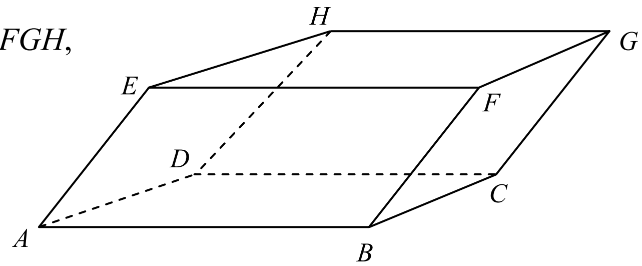
2. Calculate (a) $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \boldsymbol{.} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, (b) $(2\mathbf{i}+3\mathbf{j}-\mathbf{k})\boldsymbol{.}(3\mathbf{i}+4\mathbf{j}+7\mathbf{k})\times(7\mathbf{i}+4\mathbf{j}+2\mathbf{k})$.

3. Derive the sine rule for the triangle ABC by considering the vector product of $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$ with $\mathbf{a}$.



4. Determine whether the four points (1, –2, 1), (4, –1, 5), (3, –2, 7) and (6, 1, 1) lie in a plane or not.

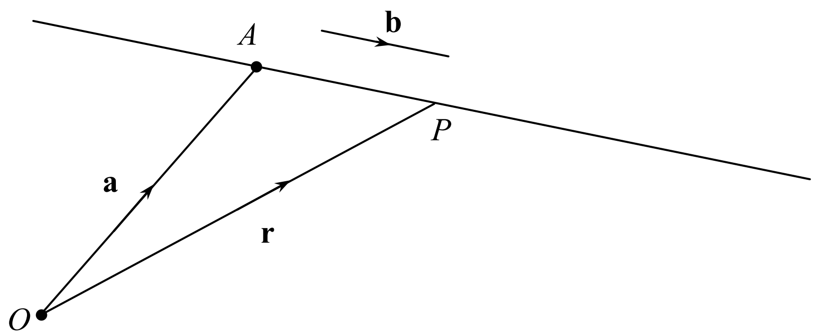
5. Find the volume of the parallelepiped $ABCDEFGH$, where $\overrightarrow{AB}=\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\overrightarrow{AD}=\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, $\overrightarrow{AE}=\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.



6. Find the volume of the tetrahedron $ABCD$ if the points A, B, C and D have coordinates (2, 1, 3), (1, –1, 2), (2, –2, 4) and (3, 1, –1), respectively.

4.4 Equations of lines

Consider the line through point **a** and parallel to the vector **b**.



For any point P on the line, with position vector **r**, $\mathbf{r}-\mathbf{a}$ is parallel to **b**. This can be used to write down different forms for the equation of the line. The vector $\overrightarrow{\mathbf{AP}}$ is a multiple of the vector **b**, $t\mathbf{b}$ say, and this gives the parametric form,

$$\mathbf{r}=\mathbf{a}+t\mathbf{b}, \text{ where } t \text{ is a parameter.}$$

The vector product form is

$$(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0.$$

This does not involve a parameter and this can be an advantage in some circumstances.

Example 4.4.1

Find the vector product form of the equation of the line through $\begin{bmatrix}1\\4\\5\end{bmatrix}$ and parallel to $\begin{bmatrix}2\\-1\\6\end{bmatrix}$.

Solution

$$\left(\mathbf{r}-\begin{bmatrix}1\\4\\5\end{bmatrix}\right)\times\begin{bmatrix}2\\-1\\6\end{bmatrix}=\mathbf{0}$$

The form given in Example 4.4.1 can be expanded

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x-1 & y-4 & z-5\\ 2 & -1 & 6\end{vmatrix}=\mathbf{0},$$

and the coefficients of **i**, **j** and **k** must all be zero.

$$6(y-4)+(z-5)=0$$
$$6(x-1)-2(z-5)=0$$
$$-(x-1)-2(y-4)=0.$$

These three equations can then be expressed in one double equation:

$$\frac{x-1}{2}=\frac{y-4}{-1}=\frac{z-5}{6}.$$

This form of the equation of the line is called **direction ratio** form because it can be obtained simply from the ratios of the x, y and z components of the direction of the line:

$$x-1:y-4:z-5=2:-1:6$$
$$\Rightarrow\frac{x-1}{2}=\frac{y-4}{-1}=\frac{z-5}{6}.$$

The equation of the line through **a** and parallel to **b** can be expressed in the forms

- parametric: $\mathbf{r}=\mathbf{a}+t\mathbf{b}$,
- vector product: $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$,
- direction ratio: $\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}$

Example 4.4.2

Find the direction ratio form of the equation of the line through $A(1,-1,4)$ and $B(2,2,3)$.

Solution

$$x-1 : y+1 : z-4 = 1 : 3 : -1$$
$$\Rightarrow \frac{x-1}{1} = \frac{y+1}{3} = \frac{z-4}{-1}.$$

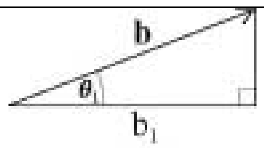
If one, or more, of the components of the vector **b** are zero, then the direction ratio form of the equation has to be modified. For example, the equation of the line through (1, 2, 3) and (3, 2, 4) could be expressed as

$$\frac{x-1}{2} = \frac{z-3}{1},\ y = 2.$$

For the line

$$\frac{x-a_1}{b_1} = \frac{x-a_2}{b_2} = \frac{x-a_3}{b_3}\ ,$$

the vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is a vector parallel to the line. Suppose that **b** makes angles θ_1, θ_2 and θ_3 with the $x-$, $y-$ and $z-$axes, respectively. Then

	$\cos\theta_i = \dfrac{b_i}{\lvert\mathbf{b}\rvert}$, for $1 \le i \le 3$.

The expressions $\dfrac{b_i}{\lvert\mathbf{b}\rvert}$ are termed the **direction cosines** of the line.

Note that if *l*, *m*, and *n*, are the direction cosines of a line, then

$$l^2 + m^2 + n^2 = \frac{1}{\lvert\mathbf{b}\rvert^2}\left(b_1^{\,2} + b_2^{\,2} + b_3^{\,2}\right) = 1.$$

The direction cosines of a line, *l*, *m*, and *n*, satisfy $l^2 + m^2 + n^2 = 1$.

Example 4.4.3

Find the angles made by the line

$$\frac{x-1}{\sqrt{2}}=\frac{x-2}{1}=\frac{x-3}{-1}$$

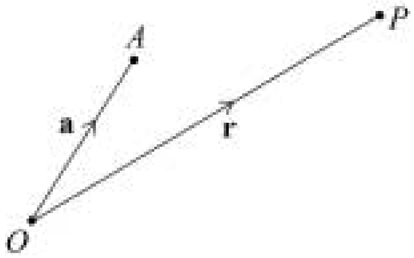
with the coordinate axes.

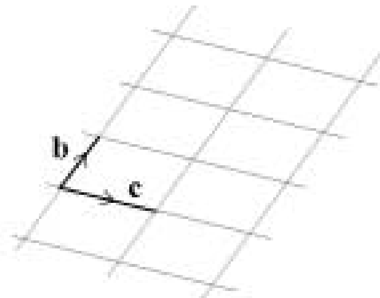
Solution

$\sqrt{2}^2+1^2+(-1)^2=4$, so the direction cosines are $\frac{\sqrt{2}}{2}$, $\frac{1}{2}$ and $\frac{-1}{2}$. The angles are $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$, $\cos^{-1}\left(\frac{1}{2}\right)$ and $\cos^{-1}\left(\frac{-1}{2}\right)$. Ie. $45°$, $60°$ and $120°$.

4.5 Equations of planes

Consider the plane through point **a** and containing the non-parallel vectors **b** and **c**.





For any point P on the plane, with position vector **r**, $\mathbf{r} - \mathbf{a}$ can be expressed as a combination of $\mathbf{b}$s and $\mathbf{c}$s, $\lambda\mathbf{b} + \mu\mathbf{c}$ say. This gives the parametric form of the equation of a plane:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$, where λ and μ are parameters.

The position vector of a general point on the plane.

Any specific point on the plane.

Any two non-parallel vectors lying on the plane.

Example 4.5.1

Find an equation of the plane through the points (1, 0, 2), (2, 1, 3), and (3, 2, -1).

Solution

$\begin{bmatrix}2\\1\\3\end{bmatrix}-\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}3\\2\\-1\end{bmatrix}-\begin{bmatrix}2\\1\\3\end{bmatrix}=\begin{bmatrix}1\\1\\-4\end{bmatrix}$. So the points of the plane are given by, for example,

$$\mathbf{r}=\begin{bmatrix}1\\0\\2\end{bmatrix}+\lambda\begin{bmatrix}1\\1\\1\end{bmatrix}+\mu\begin{bmatrix}1\\1\\-4\end{bmatrix}$$

For some purposes it is best to have a form of equation of the plane which contains no parameters. Consider a plane with the equation

$$\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$$

and suppose we could find a vector $\mathbf{n}$ perpendicular to the plane. Then $\mathbf{b}.\mathbf{n}=\mathbf{c}.\mathbf{n}=0$ and so

$$\mathbf{r}.\mathbf{n}=\mathbf{a}.\mathbf{n}+\lambda\mathbf{b}.\mathbf{n}+\mu\mathbf{c}.\mathbf{n}=\mathbf{a}.\mathbf{n}\ .$$

Thus $\mathbf{r}.\mathbf{n}$ is the same for **all** points **r** on the plane.

To find the Cartesian equation of a plane in the form $\mathbf{r}.\mathbf{n}=d$, it is necessary to find the vector **n** which is normal to the plane. The vector product can be used for this.

Example 4.5.2

Find the equation of the plane containing the two lines $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution

The normal to the plane is in the direction

$$\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

Then $\mathbf{r}.\mathbf{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = 3x - y + 2z$

$3x - y + 2z = d$ is satisfied by (2, 1, 1) and (3, 0, –1) if $d = 7$, so the equation is

$$3x - y + 2z = 7.$$

Cartesian equations of planes have the form

$$ax + by + cz = d\ ,$$

where a, b, c and d are constants and the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is perpendicular to the plane.

Vector products can be used similarly in finding the equation of the line of intersection of two non-parallel planes because the direction of this line is perpendicular to the two normals.

Example 4.5.3

Find the equation of the line of intersection of the two planes $x+y-z=7$ and $2x+3y-4z=2$.

Solution

The direction of the common line is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 3 & -4 \end{vmatrix} = -\mathbf{i}+2\mathbf{j}+\mathbf{k}.$$

To find any common point, set z equal to zero (for example) and then

$$x+y=7, \quad 2x+3y=2$$
$$\Rightarrow y=-12, x=19.$$

The line is $\mathbf{r}=\begin{bmatrix} 19 \\ -12 \\ 0 \end{bmatrix}+\lambda\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

In the example above, setting z equal to zero was a simple device to reduce the number of variables in the equations. In general, any one of x, y or z could be given any value you wish. However, if the common line is perpendicular to one or more of the axes, then some choices will not work. This is illustrated in the next example.

Example 4.5.4

Find the equation of the line of intersection of the two planes $x+y-z=7$ and $2x+2y-3z=2$.

Solution

The direction of the common line is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 2 & -3 \end{vmatrix} = -\mathbf{i}+\mathbf{j}.$$

Setting $z=0$ leads to the inconsistent equations

$$x+y=7, \quad 2x+2y=2 \left(\text{i.e. } x+y=1\right).$$

So, set x equal to zero instead,

$$y-z=7, \quad 2y-3z=2$$
$$\Rightarrow y=19, z=12.$$

The line is $\mathbf{r}=\begin{bmatrix} 0 \\ 19 \\ 12 \end{bmatrix}+\lambda\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Finding the point of intersection of a line and a plane is usually simply a matter of finding the appropriate value for the parameter of the line. This is illustrated in the next example.

Example 4.5.5

Find the point of intersection of the line

$$\mathbf{r} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ with the plane } 3x - y + 2z = 5.$$

Solution

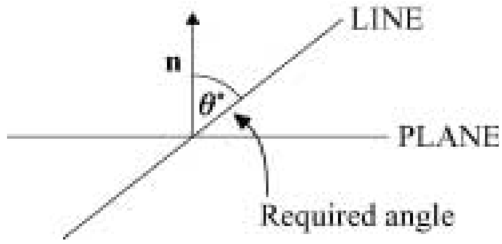
Substituting for $x, y,$ and z,

$$3(1-t) - (-2+t) + 2(3+t) = 5$$
$$\Rightarrow 11 - 2t = 5$$
$$\Rightarrow t = 3$$

The point of intersection is $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}$.

4.6 Angles between lines and planes

To find the angle between a line and a plane, first find the acute angle $\theta°$ between the direction of the line and a perpendicular to the plane. The required angle is then $90-\theta°$.



Example 4.6.1

Find the angle between the line

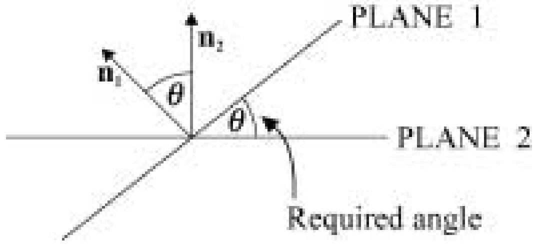
$$\mathbf{r}=\begin{bmatrix}2\\1\\1\end{bmatrix}+t\begin{bmatrix}1\\2\\-1\end{bmatrix} \text{ and the plane } x-y+2z=7.$$

Solution

$$\begin{bmatrix}1\\2\\-1\end{bmatrix}\bullet\begin{bmatrix}1\\-1\\2\end{bmatrix}=1-2-2$$
$$\Rightarrow \sqrt{6}\sqrt{6}\cos\theta=-3$$
$$\Rightarrow \cos\theta=-\frac{1}{2}$$
$$\Rightarrow \theta=120°$$

The acute angle between the line and the perpendicular to the plane is $60°$. The required angle is $90°-60°=30°$.

To find the angle between two planes, simply find the acute angle between their perpendiculars.



Example 4.6.2

Find the acute angle between the planes

$$3x - 5y - 4z = 8 \qquad \text{and} \qquad 3x - 4z = -7.$$

Solution

$$\begin{bmatrix} 3 \\ -5 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = 25$$

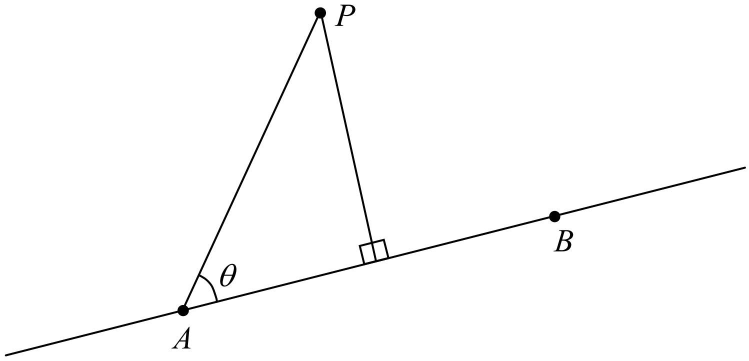
$$\Rightarrow \sqrt{50}\sqrt{25}\cos\theta = 25$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = 45°.$$

4.7 Shortest distances

The perpendicular distance from the point P to the line AB is $\overrightarrow{AP}\sin\theta$.



This distance is therefore given by

$$\frac{\left|\overrightarrow{AP}\right|\left|\overrightarrow{AB}\right|\sin\theta}{\left|\overrightarrow{AB}\right|}=\frac{\left|\overrightarrow{AP}\times\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|}.$$

> The perpendicular distance from the point P to a line through the points A and B is
>
> $$\frac{\left|\overrightarrow{AP}\times\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|}$$

Example 4.7.1

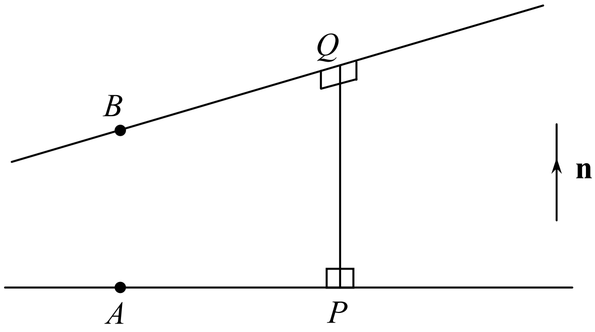
Find the perpendicular distance from the point $P(2, -1, 3)$ to the straight line with the equation $x-2=\frac{y+2}{3}=\frac{z-1}{2}$.

Solution

The line has direction $\begin{bmatrix}1\\3\\2\end{bmatrix}$ and $(2, -2, 1)$ is a point on the line. The distance is therefore

$$\frac{\left|\begin{bmatrix}0\\1\\2\end{bmatrix}\times\begin{bmatrix}1\\3\\2\end{bmatrix}\right|}{\sqrt{1^2+3^2+2^2}}=\frac{\left|\begin{bmatrix}-4\\2\\-1\end{bmatrix}\right|}{\sqrt{14}}=\sqrt{\frac{3}{2}}.$$

The shortest distance between two lines is along a line which is perpendicular to both lines. The direction of this perpendicular can be found using the vector product of the direction vectors of the two lines. Consider two lines AP and BQ, where PQ is perpendicular to both lines and is in the direction of the vector **n**.



Then $\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PQ} + \overrightarrow{QB}$.

And so $\left|\overrightarrow{AB}.\mathbf{n}\right| = \left|\overrightarrow{AP}.\mathbf{n} + \overrightarrow{PQ}.\mathbf{n} + \overrightarrow{QB}.\mathbf{n}\right|$

$\Rightarrow \left|\overrightarrow{AB}.\mathbf{n}\right| = \left|\overrightarrow{PQ}\right|\left|\mathbf{n}\right|$.

The shortest distance, $\left|\overrightarrow{PQ}\right|$, is therefore equal to $\dfrac{\left|\overrightarrow{AB}.\mathbf{n}\right|}{|\mathbf{n}|}$.

> To find the shortest distance between two lines:
>
> - find any vector $\overrightarrow{AB}$ from a point on one line to a point on the other
> - find a vector **n** perpendicular to both lines
> - calculate $\dfrac{\left|\overrightarrow{AB}.\mathbf{n}\right|}{|\mathbf{n}|}$

Example 4.7.2

Find the shortest distance between the lines $\mathbf{r} = \begin{bmatrix}1\\0\\2\end{bmatrix} + \lambda\begin{bmatrix}2\\-1\\5\end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix}1\\1\\0\end{bmatrix} + \mu\begin{bmatrix}1\\0\\2\end{bmatrix}$.

Solution

$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 5\\ 1 & 0 & 2\end{vmatrix} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ is perpendicular to both lines.

A vector between the lines is $\mathbf{j} - 2\mathbf{k}$.

Shortest distance $= \dfrac{|1-2|}{\sqrt{4+1+1}} = \dfrac{1}{6}\sqrt{6}$.

Exercise 4B

1. Find the equation of the line of intersection of the planes $x + y - 2z = 5$ and $x + 2y + z = 7$.

2. Find the distance of the point (1, 1, 2) from the line $\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

3. A line has the equation
$$\frac{x-1}{-2} = \frac{y-3}{4}, \quad z = 1.$$
Express this in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

4. Find the Cartesian equation of the plane containing the point (1, –1, 4) and the line with equation
$$\frac{x-2}{1} = \frac{y-3}{2} = \frac{z-1}{-1}.$$

5. Find the Cartesian equation of the plane through the points (1, 2, 3), (2, –1, –1) and (3, 4, 6).

6. Four points are given by *A* (1, –2, 0), *B* (3, –3, –1), *C* (2, 3, –1) and *D* (3, 4, –5).

   (a) Calculate $\overrightarrow{AB} \times \overrightarrow{CD}$.

   (b) Hence find the shortest distance between *AB* and *CD*.

7. Consider the plane Π_1 and the lines L_1 and L_2 given by
$$\Pi_1 \quad x + y - 5z = 3,$$
$$L_1 \quad \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{-8},$$
$$L_2 \quad \frac{x+1}{1} = \frac{y}{2}, \quad z = 7.$$

   (a) Show that L_1 and L_2 are coplanar.

   (b) Find the equation of the plane Π_2, which contains L_1 and L_2.

   (c) Find the equation of the line of intersection of planes Π_1 and Π_2.

Miscellaneous exercises 4

1. Given that $\mathbf{a}$ is a non-zero vector and that
$$\mathbf{a} \times \mathbf{x} = \mathbf{x} \times \mathbf{a},$$
state what conclusions may be drawn about the vector $\mathbf{x}$. [JMB, 1980]

2. The points A, B and R have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$, respectively; A and B are fixed points and R varies. Describe, geometrically, the locus of R in each of the cases
(a) $(\mathbf{r} - \mathbf{a}).\mathbf{b} = 0$,
(b) $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. [JMB, 1978]

3. Find the distance of the point (2, 4, 7) from the plane determined by the points (3, 4, 4), (4, 5, 7) and (6, 8, 9).
[JMB, 1970]

4. (a) Three non-collinear points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ relative to an origin O, which does not lie in the plane ABC. Given that P is a variable point with position vector $\mathbf{r}$ relative to O, show that the equation $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ represents a line through A in the direction of the vector $\mathbf{b}$.

(b) Find a vector equation for each of the planes
(i) through C and perpendicular to the line $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$,
(ii) through C and containing the line $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. [JMB, 1979]

5. No three of the points A, B, C and D are collinear and they are such that
$$\overrightarrow{AB} \times \overrightarrow{AC} = \overrightarrow{AC} \times \overrightarrow{AD}.$$
Prove that
(i) A, B, C and D are coplanar,
(ii) AC bisects BD. [JMB, 1974]

6. The locus of P is the line whose vector equation is
$$\mathbf{r} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$
The perpendicular from P to the plane
$$x + 2y - z = 12$$
meets it at Q. Find a vector equation for the locus of Q.
[JMB, 1977]

7. The points A (1, 0, –2), B (2, –2, 1) and C (5, –4, 0) have position vectors **a**, **b** and **c**, respectively.

   (a) Find the vector product of $\overrightarrow{AC}$ and $\overrightarrow{AB}$.

   (b) Hence, or otherwise, find

   (i) an equation for the plane ABC,

   (ii) the area of triangle ABC.

   (c) Find an equation for the plane which passes through A and which is perpendicular to the plane ABC and to the plane $(\mathbf{r}-\mathbf{a}).\mathbf{b}=0$.

[JMB,1978]

8. (a) Prove that $\left|\overrightarrow{PQ}\times\overrightarrow{QR}\right|$ is twice the area of the triangle PQR.

   (b) The points P, Q and R are (1, 3, 2), (4, 5, 1) and (3, 3, 1), respectively.

   (i) Find the Cartesian equation of the plane PQR.

   (ii) Find the coordinates of the foot N of the perpendicular from the origin O to the plane PQR.

   (iii) Find the volume of the tetrahedron $OPQR$. [JMB, 1976]

9. The equation of the plane Π is

$$3x+4y-z+9=0.$$

   (a) (i) Write down a vector equation, in terms of a parameter t, for the line through the point A (2, 3, 1) and at right angles to the plane Π.

   (ii) Find the coordinates of the point B where this line meets the plane Π.

   (b) The point C lies on Π and has coordinates (–6, 3, 3). Find

   (i) the Cartesian equation of the plane ABC,

   (ii) the coordinates of the reflection D of the point A in Π,

   (ii) a vector equation for the reflection of the line AC in Π.

[JMB,1977]

10. Relative to the origin O, the points A, B, C and D have position vectors

$$\mathbf{i}+\mathbf{j},\quad \mathbf{i}+\tfrac{1}{2}\mathbf{j}+\mathbf{k},\quad \mathbf{j}+\mathbf{k}\ \text{ and }\ \mathbf{i}+\mathbf{j}+\mathbf{k},$$

   respectively.

   (a) Calculate

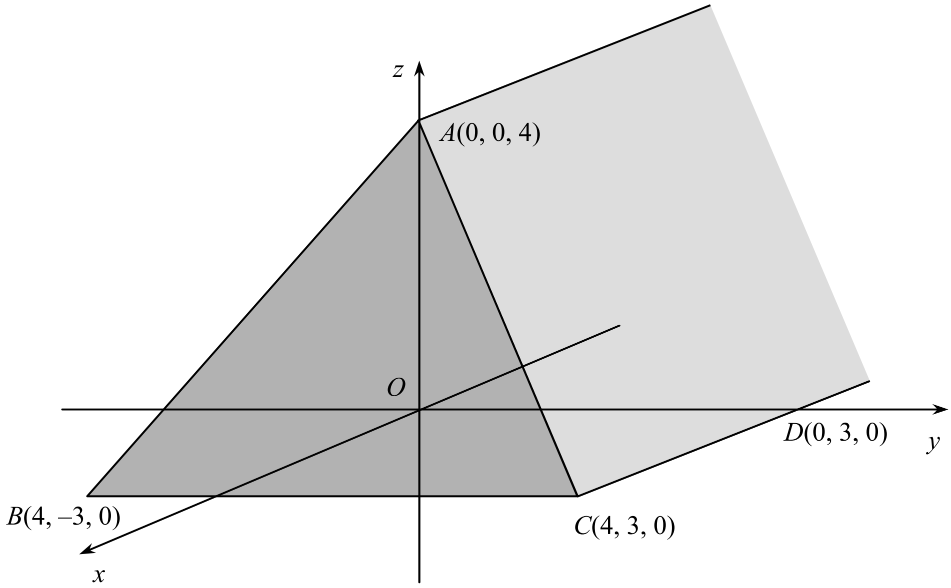
   (i) the angle between **j** and the plane ABC,

   (ii) the volume of the pyramid $ABCD$.

   (b) (i) Show that the distance of O from the plane ABC is $\dfrac{3}{\sqrt{6}}$ and find the volume of the pyramid $OABC$.

   (ii) Deduce, or find otherwise, the ratio OP:PD, where P is the point where OD meets the plane ABC. [JMB, 1975]

11. The three vertices A, B and C of the sloping triangular end of a roof are at (0, 0, 4), (4, –3, 0) and (4, 3, 0), as shown in the diagram below.



(a) Find

(i) a vector perpendicular to the plane ABC,

(ii) the equation of the plane ABC in the form $\mathbf{r.n} = d$.

(b) Find the perpendicular distance from the origin O to the ABC.

(c) The point D on the roof has coordinates (0, 3, 0). Find, to the nearest degree, the angle inside the roof between the planes ABC and ACD.

[AQA-NEAB, 2000]

12. The lines L_1 and L_2 have vector equations $\mathbf{r} = (2+\lambda)\mathbf{i} + (-2-\lambda)\mathbf{j} + (7+\lambda)\mathbf{k}$ and $\mathbf{r} = (4+4\mu)\mathbf{i} + (26+14\mu)\mathbf{j} + (-3-5\mu)\mathbf{k}$, respectively, where λ and μ are scalar parameters.

(a) The vector $\mathbf{n} = -\mathbf{i} + a\mathbf{j} + b\mathbf{k}$, where a and b are integers, is perpendicular to both L_1 and L_2. Find the value of a and the value of b.

(b) The point P on L_1 and the point Q on L_2 are such that $\overrightarrow{PQ} = m\mathbf{n}$ for some scalar constant m.

(i) Determine the value of m.

(ii) Deduce the shortest distance between L_1 and L_2.

[AQA-NEAB, 2000]

13. The line L passes through the point $A(4, 4, -3)$ and has vector equation

$$\mathbf{r} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

(a) Show that the line M which passes through the points $B(4, 6, 1)$ and $C(6, 7, 3)$ is parallel to the line L.

(b) (i) Given that angle ACB is θ, show that $\cos\theta = \dfrac{19}{21}$.

(ii) Express $\sin\theta$ in surd form.

(c) Hence, or otherwise, show that the shortest distance between the lines L and M is $k\sqrt{5}$, where k is a rational number to be determined.

[AQA-NEAB, 2001]

14. Two skew lines, L_1 and L_2, have vector equations

$$\mathbf{r} = \begin{bmatrix} -4 \\ 0 \\ 8 \end{bmatrix} + \lambda\begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix} \text{ and } \mathbf{r} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} + \mu\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \text{ respectively.}$$

The plane Π contains L_1 and intersects L_2 at the point $(-1, 3, -2)$.

(a) Show that the vector $\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ is normal to the plane Π.

(b) Hence, or otherwise, find the equation of the plane in the form $\mathbf{r.n} = d$.

[AQA-NEAB, 2001]

Chapter 5: Inverse Matrices

This chapter introduces the idea of the inverse of a matrix. When you have completed it, you will:

- know what is meant by an inverse;
- be able to find the inverses of 2×2 and 3×3 matrices;
- know what is meant by singularity;
- understand why $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$;
- understand why $\left|\mathbf{A}^{-1}\right| = \dfrac{1}{|\mathbf{A}|}$.

5.1 Inverses

For any transformation T, the transformation which reverses the effect of T is called the **inverse** of T.

The inverses of some transformations are self-evident. For example:

Transformation	Inverse
Rotation of θ about an axis	Rotation of $-\theta$ about the same axis
Reflection in a line or plane	Reflection in the same line or plane
Enlargement, $\times k$	Enlargement, $\times \frac{1}{k}$

Transformations which are their own inverses are called **self-inverse**. Thus, all reflections are self-inverse, as are all rotations of 180°.

Suppose the matrices of two inverse transformations are **A** and **B**. Then the transformations represented by **AB** and **BA** leave all points unchanged and therefore $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$.

This gives us the definition of an inverse matrix.

Let **A** be any square matrix. The inverse of **A**, denoted by $\mathbf{A}^{-1}$, is a matrix such that

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

As you will see in Section 5.2, not all matrices have inverses.

5.2 2 × 2 Matrices

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any 2×2 matrix. Consider the matrix formed by switching the elements on the main diagonal and reversing the signs of the other elements, i.e. $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\mathbf{I},$$

and $$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)\mathbf{I}$$

Providing that $ad-bc \neq 0$, you can divide by $ad-bc$ to obtain the following result:

> The inverse of $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for $ad-bc \neq 0$,
>
> is $\mathbf{M}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The quantity $ad-bc$ in the above result is $|\mathbf{M}|$. You now have a formula for $\mathbf{M}^{-1}$ in the case where $|\mathbf{M}| \neq 0$.

If $|\mathbf{M}| = 0$, then $\mathbf{M}$ cannot have an inverse. A simple proof of this is of the type known as **proof by contradiction**. Suppose that $\mathbf{M}^{-1}$ exists. Then $\mathbf{I} = \mathbf{M}\mathbf{M}^{-1}$ and, taking determinants,

$$\begin{aligned} 1 &= |\mathbf{M}\mathbf{M}^{-1}| \\ &= |\mathbf{M}||\mathbf{M}^{-1}| \\ &= 0 \times |\mathbf{M}^{-1}| \\ &= 0. \end{aligned}$$

This contradiction proves that $\mathbf{M}^{-1}$ does not exist.
This proof works for a square matrix of any size, and so gives a general conditon for a matrix to not have an inverse.

> A square matrix which has no inverse is called **singular**
> All matrices with determinant zero are singular

The idea used in the proof also gives a general result for the determinant of an inverse matrix:

> For any matrix $\mathbf{M}$ with inverse $\mathbf{M}^{-1}$,
>
> $$|\mathbf{M}^{-1}| = \frac{1}{|\mathbf{M}|}$$

5.3 3 × 3 Matrices

In Section 5.2 you saw the importance of the determinant in the existence, or otherwise, of the inverse of a general square matrix. The determinant was also used in finding the inverse of a 2×2 matrix.

Determinants are even more obviously used when finding the inverse of a 3×3 matrix. As well as the determinant of the whole matrix, so-called **minor determinants** are also used. For example, for the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix},$$

the minor determinant corresponding to each element is the determinant of the 2×2 matrix formed by deleting the row and column containing that element. Thus

$$\begin{vmatrix} 0 & 5 \\ 7 & 8 \end{vmatrix} = -35 \quad \text{corresponds to 1,}$$

$$\begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} = -10 \quad \text{corresponds to 0, etc.}$$

The matrix of these minors is then given by $\begin{bmatrix} -35 & 2 & 28 \\ -5 & -10 & -5 \\ 10 & -7 & -8 \end{bmatrix}$.

The full algorithm (sequence of steps) for finding the inverse of a 3×3 matrix **M** is:

1. Find the determinant of **M**. If this is zero then stop.
2. Find the matrix of minor determinants.
3. Alter the signs of the minor determinants in the positions marked with minus signs: ⟶ $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$
   This new matrix is called the matrix of **cofactors**.
4. Transpose the matrix of cofactors.
5. Divide by $|\mathbf{M}|$.

Example 5.3.1

Find the inverse of $\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}$.

Solution

Step 1: $|\mathbf{M}| = 1(0-35) - 2(32-30) + 3(28-0) = 45$

Step 2: Matrix of minors = $\begin{bmatrix} -35 & 2 & 28 \\ -5 & -10 & -5 \\ 10 & -7 & -8 \end{bmatrix}$

Step 3: Matrix of cofactors = $\begin{bmatrix} -35 & -2 & 28 \\ 5 & -10 & 5 \\ 10 & 7 & -8 \end{bmatrix}$

Steps 4&5: $\mathbf{M}^{-1} = \dfrac{1}{45}\begin{bmatrix} -35 & 5 & 10 \\ -2 & -10 & 7 \\ 28 & 5 & -8 \end{bmatrix}$

Check: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 6 & 7 & 8 \end{bmatrix}\begin{bmatrix} -35 & 5 & 10 \\ -2 & -10 & 7 \\ 28 & 5 & -8 \end{bmatrix} = \begin{bmatrix} 45 & 0 & 0 \\ 0 & 45 & 0 \\ 0 & 0 & 45 \end{bmatrix}$, as required.

Example 5.3.2

Find the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -3 \\ -1 & 0 & 5 \end{bmatrix}$.

Solution

First, calculate $|\mathbf{A}| = 2(10-0) - 1(5-3) - 1(0+2) = 16$.

Next, calculate the minor determinants: $\begin{bmatrix} 10 & 2 & 2 \\ 5 & 9 & 1 \\ -1 & -5 & 3 \end{bmatrix}$,

and form the matrix of cofactors: $\begin{bmatrix} 10 & -2 & 2 \\ -5 & 9 & -1 \\ -1 & 5 & 3 \end{bmatrix}$.

Transposing and dividing by the determinant gives $\mathbf{A}^{-1} = \dfrac{1}{16}\begin{bmatrix} 10 & -5 & -1 \\ -2 & 9 & 5 \\ 2 & -1 & 3 \end{bmatrix}$.

Exercise 5A

1. Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} 1 & 0 \\ 5 & 7 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 4 & -3 \\ -2 & -1 \end{bmatrix}$.

2. Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 1 & -7 \\ 0 & -3 & 6 \\ -1 & 1 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & -2 & 4 \\ 2 & 5 & -1 \\ -3 & 4 & 1 \end{bmatrix}$.

3. Determine which of the following matrices are singular.

(a) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 15 \\ 1 & 5 \end{bmatrix}$, (c) $\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 15 & 11 \\ 23 & -6 & 19 \end{bmatrix}$, (e) $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 8 & 19 & 26 \end{bmatrix}$, (f) $\begin{bmatrix} 3 & -18 & 2 \\ -1 & 13 & -3 \\ 1 & 8 & -4 \end{bmatrix}$.

4. For which value of k is the following matrix singular?

$$\begin{bmatrix} 1 & 0 & 2 \\ 5 & 1 & k \\ 6 & -1 & 1 \end{bmatrix}$$

5. For $k \neq \frac{5}{2}$, find the inverse of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 3 & 1 \\ k & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$.

6. (a) Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}$.

(b) The image of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ when transformed by $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$.

Find x, y and z.

5.4 Products

It can be useful to have a way of finding the inverse of a product of matrices without first having to work out the product.

Consider the product $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$.

$$\begin{aligned}(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) &= \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B}, \quad \text{by associativity}\\ &= \mathbf{B}^{-1}\mathbf{I}\,\mathbf{B}\\ &= \mathbf{B}^{-1}\mathbf{B}\\ &= \mathbf{I}.\end{aligned}$$

Therefore $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of $\mathbf{AB}$.

This idea enables you to find the inverse of any product of square matrices of the same size.

> Suppose that **A**, **B**, **C**, … are square matrices of the same size. If **A**, **B**, **C**, … all have inverses, then so does the product **ABC**… . Furthermore,
> $$(\mathbf{ABC}\ldots)^{-1} = \ldots\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

Example 5.4.1

(a) Given that $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ and $\mathbf{AB}$.

(b) Hence find $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & 6 \end{bmatrix}^{-1}$.

Solution

(a) $\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$, $\mathbf{B}^{-1} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{AB} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 0 & 2 & 6 \end{bmatrix}$.

(b) $$\begin{aligned}(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} &= \frac{1}{12}\begin{bmatrix} 6 & -3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}\\ &= \frac{1}{6}\begin{bmatrix} 12 & -3 & 0 \\ -6 & 3 & 0 \\ 2 & -1 & 1 \end{bmatrix}.\end{aligned}$$

Miscellaneous Exercises 5

1. The matrix **A** is defined by $\mathbf{A}=\begin{bmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{bmatrix}$.

   (a) Given that $\mathbf{A}^{-1}$ exists, show that $a \neq -b$.

   (b) Find $\mathbf{A}^{-1}$ in terms of a and b.

[AQA–NEAB, 2000]

2. The image of the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ when transformed by the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ is the vector $\begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix}$. Use the inverse matrix to find a, b and c.

3. Find the values of k for which the matrix $\begin{bmatrix} k+2 & 2k+3 & 0 \\ -4 & k-5 & 2-2k \\ 3 & 4 & k-1 \end{bmatrix}$ has no inverse.

[JMB, 1978]

4. (a) Find matrices **A**, **B** and **C** such that $\mathbf{AB}=\mathbf{AC}$ but $\mathbf{B}\neq\mathbf{C}$.

   (b) If $\mathbf{A}^{-1}$ exists, show that $\mathbf{AB}=\mathbf{AC} \Rightarrow \mathbf{B}=\mathbf{C}$.

5. For a given matrix **M**, three vectors **a**, **b** and **c** are such that

$$\mathbf{Ma}=\mathbf{b},\ \ \mathbf{Mb}=\mathbf{c}\ \text{ and }\ \mathbf{Mc}=\mathbf{a}+2\mathbf{b}+3\mathbf{c}.$$

   (a) State the results of $\mathbf{M}^{-1}\mathbf{b}$ and $\mathbf{M}^{-1}\mathbf{c}$.

   (b) Hence find $\mathbf{M}^{-1}\mathbf{a}$.

6. The matrix **A** is given by $\mathbf{A}=\begin{bmatrix} k & 5 & 4 \\ 3 & k & 4 \\ 1 & 1 & 1 \end{bmatrix}$.

   (a) Show that **A** is **not** singular, whatever the value of the real constant k.

   (b) Find $\mathbf{A}^{-1}$ in terms of k.

   (c) Given that $k=4$, find the point mapped on to (9, 6, 2) by the transformation represented by **A**.

7. Two matrices **A** and **B** are such that

$$\mathbf{A}^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } \mathbf{B}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}.$$

Find the matrix **C** such that $\mathbf{ABC} = \mathbf{I}$.

8. The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Show that $\mathbf{A}^3 = 7\mathbf{A}^2 - 14\mathbf{A} + 8\mathbf{I}$.

(b) Deduce that $\mathbf{A}^{-1} = \frac{1}{8}\left(\mathbf{A}^2 - 7\mathbf{A} + 14\mathbf{I}\right)$.

Further questions on inverse matrices are given in Chapter 7.

Chapter 6: Solving Linear Equations

This chapter considers the solution of systems of three equations in three unknowns, such as

$$\begin{aligned} 3x + y - 2z &= 7 \\ 2x + 4y + z &= 8 \\ 5x - y + 3z &= 9. \end{aligned}$$

When you have completed it, you will:

- know how to interpret such equations geometrically;
- understand the connection with invertible and singular matrices;
- be able to solve such systems of equations;
- understand the connection with linear independence of vectors.

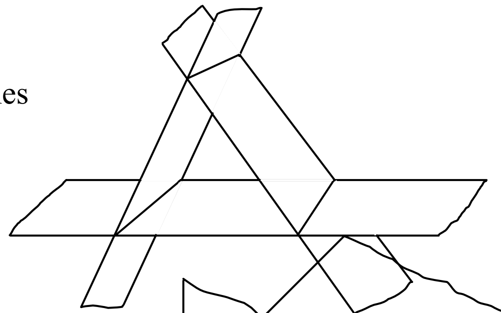
6.1 Geometrical interpretation

An equation such as

$$3x + y - 2z = 7$$

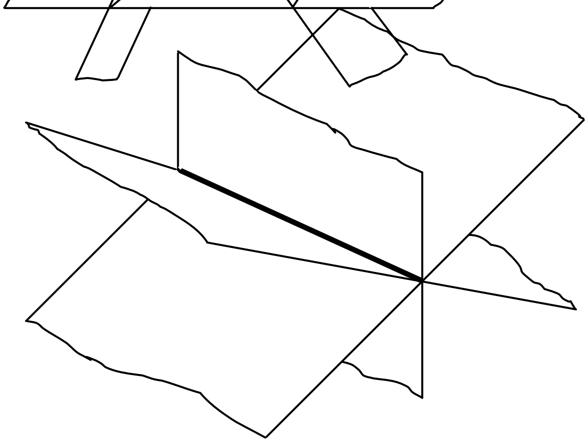
can be thought of as the equation of a plane. When there are three such planes, there are several possibilities:

- two of the planes are the same
- two of the planes are parallel
- the three planes form a triangular prism



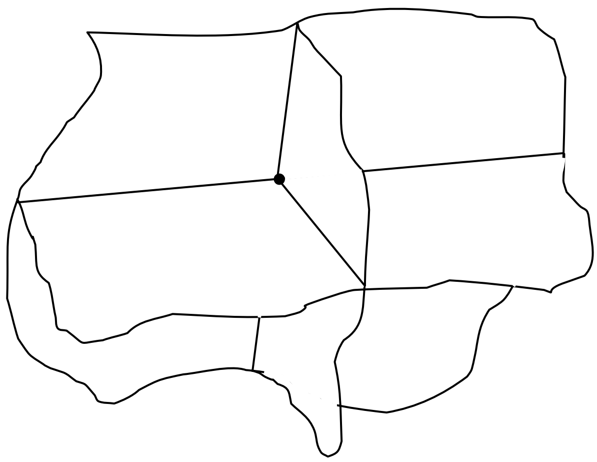
  - there are no points common to all three planes
  - the equations have no solutions
  - they are **inconsistent**

- the three planes form a sheaf



  - there is a common line
  - the equations have infinitely many solutions

- the three planes intersect at a unique point



  - the equations have precisely one solution

The first two of these cases can be spotted from the equations immediately. For example,

$$3x + 2y - z = 7 \qquad [1]$$
$$6x + 4y - 2z = 14 \qquad [2]$$
$$9x + 6y - 3z = 18 \qquad [3]$$

Equation 2 is a multiple of equation 1; therefore they represent the **same** plane.

Equation 3 is a multiple of equation 1 except for the constant term; therefore these equations represent **parallel** planes.

When considering a system of three equations, first check that no two planes are the same or parallel.

Then there are just three possibilities:

- prism — no solutions
- sheaf — infinitely many solutions
- unique point — one solution

6.2 Inverse matrices

A system of three equations, such as those given at the beginning of this chapter, can be considered to be a single matrix equation. For example,

$$\begin{bmatrix} 3 & 1 & -2 \\ 2 & 4 & 1 \\ 5 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

In general, the matrix equation will be of the form $\mathbf{Ar} = \mathbf{b}$, where $\mathbf{A}$ is a 3×3 matrix.

If **A** is invertible, then

$$\mathbf{A}^{-1}(\mathbf{Ar}) = \mathbf{A}^{-1}\mathbf{b}$$

$$\Rightarrow \mathbf{r} = \mathbf{A}^{-1}\mathbf{b}.$$

This method gives a unique solution and so it can only be used in cases where the three planes have a single point of intersection.

Example 6.2.1

Solve $\begin{bmatrix} 3 & 1 & -2 \\ 2 & 4 & 1 \\ 5 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 3 & 1 & -2 \\ 2 & 4 & 1 \\ 5 & -1 & 3 \end{bmatrix}^{-1} = \frac{1}{82}\begin{bmatrix} 13 & -1 & 9 \\ -1 & 19 & -7 \\ -22 & 8 & 10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{82}\begin{bmatrix} 13 & -1 & 9 \\ -1 & 19 & -7 \\ -22 & 8 & 10 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$$= \frac{1}{82}\begin{bmatrix} 164 \\ 82 \\ 0 \end{bmatrix}.$$

$$\Rightarrow x = 2, \quad y = 1, \quad z = 0.$$

Exercise 6A

1. (a) If $a, b, c \in \mathbb{R}$, show that the equations

$$x + \lambda y + z = 2a$$
$$x + y + \lambda z = 2b$$
$$\lambda x + y + \lambda z = 2c$$

have a unique solution for x, y, z provided that $\lambda \neq 1$ and $\lambda \neq -1$.

(b) In the case when $\lambda = 1$, state the condition to be satisfied by a, b and c for the equations to be consistent.

(c) In the case when $\lambda = -1$, show that $a + c = 0$ for the equations to be consistent, and solve the equations in this case.

(d) Give a geometrical description of the configuration of the three planes represented by the equations in the cases
 (i) $\lambda = -1$ and $a + c = 0$,
 (ii) $\lambda = -1$ and $a + c \neq 0$.

[NEAB]

2. (a) Find the inverse of the matrix $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 4 & -3 \\ 1 & -1 & 1 \end{bmatrix}$.

(b) Hence solve the equations

$$5x + 3y + 2z = 1$$
$$4y - 3z = 4$$
$$x - y + z = 1.$$

[JMB, 1971]

3. Find the value of λ such that the equations

$$x + y - 3z = 1$$
$$3x - y - z = 7$$
$$5x - 3y + \lambda z = 13$$

do **not** have a unique solution.

[JMB, 1977]

4. (a) Given that $\mathbf{M} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -k \\ 1 & -k & -1 \end{bmatrix}$, find $\det\mathbf{M}$ in terms of k.

(b) Determine the values of k for which the simultaneous equations

$$x + y - z = 1$$
$$x + 2y - kz = 0$$
$$x - ky - z = 1$$

have a unique solution.

(c) Solve these equations in the case when $k = 2$.

[NEAB]

6.3 Row operations

Consider the equations

$$3x + y - 2z = 7 \qquad [1]$$
$$2x + 4y + z = 8 \qquad [2]$$
$$5x - y + 3z = 9 \qquad [3]$$

You can choose to operate on these equations to make them simpler. For example, adding equations 1 and 3,

$$8x + z = 16 \qquad [4]$$

multiplying equation 3 by 4 and adding the result to equation 2,

$$22x + 13z = 44 \qquad [5]$$

As a result of these operations, y has been eliminated. The same idea could then be used again: multiplying equation 4 by 13 and subtracting the result from equation 5,

$$-82x = -164 \qquad [6]$$
$$\Rightarrow x = 2.$$

This value for x can be substituted back into equation 4 to obtain $z = 0$. Then both x and z can be substituted into equation 1 to obtain $y = 1$.

This process of simplifying equations by means of row operations depends on two simple ideas.

- Equations can be multiplied by any non-zero number
- Any multiple of a row can be subtracted from or added to another row

Before beginning row operations, you should always be clear about what you want to achieve.

- Perform row operations to reduce to two equations in two unknowns, and then to one equation in one unknown

Example 6.3.1

Solve $x-3y+4z=4$ [1]

$2x-y+3z=8$ [2]

$3x+y+2z=12$ [3]

Solution

To eliminate y:

- multiply equation 3 by 3 and add the result to equation 1,

$$10x+10z=40;$$

- add equations 2 and 3,

$$5x+5z=20.$$

Setting $z = t$, a parameter, gives $x=4-t$. Substituting in equation 3,

$$12-3t+y+2t=12$$
$$\Rightarrow y=t.$$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, a common line.

Example 6.3.2

Solve $x-3y+4z=4$ [1]

$2x-y+3z=8$ [2]

$3x+y+2z=10$ [3]

Solution

To eliminate y:

- multiply equation 3 by 3 and add the result to equation 1,

$10x+10z=34$ [4]

- add equations 2 and 3,

$5x+5z=18$ [5]

Then multiplying equation 5 by 2 and subtracting the result from equation 4 gives $0=-2$.
This is impossible. There are no solutions – the equations are inconsistent.

6.4 Linear independence

When a vector is a combination of other vectors, it is said to be **linearly dependent** on them. For example,

$$\begin{bmatrix}2\\4\\6\end{bmatrix}=2\begin{bmatrix}1\\0\\-1\end{bmatrix}+4\begin{bmatrix}0\\1\\2\end{bmatrix}$$

and so $\begin{bmatrix}2\\4\\6\end{bmatrix}$, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\2\end{bmatrix}$ are linearly dependent.

If there is no such relationship between the vectors then they are said to be **linearly independent**. One method of proving that vectors are linearly independent is shown in Example 6.4.1.

Example 6.4.1

Prove that $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ are linearly independent.

Solution

Suppose the vectors are related. Then the relationship can be written in the form

$$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}+c\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow a+c=0,\quad b=0,\quad b+c=0$$

$$\Rightarrow a=b=c=0.$$

So the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\end{bmatrix}$ are linearly independent.

The linear independence of vectors is closely related to the singularity of matrices. It is possible to prove the following result for 3-dimensional vectors

Three vectors $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ are linearly dependent if, and only if,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This gives a quick method of tackling problems such as the one already solved in Example 6.4.1.

Example 6.4.2

Prove that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent.

Solution

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0.$$

Hence they are independent.

When vectors **are** linearly dependent, then finding the actual linear dependence is a case of solving equations.

Example 6.4.3

Find α, β and γ such that

$$\begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Solution

$$\begin{bmatrix} 1 & 3 & 0 \\ -1 & 0 & 1 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

$$\Rightarrow \beta = -1, \quad \gamma = 3, \quad \alpha = 2.$$

Exercise 6B

1. (a) Find the value of p for which the equations

$$3x - y + 2z = 5$$
$$2x + y + 3z = 0$$
$$x - 3y + pz = 7$$

do **not** have a unique solution.

(b) Show that the vectors

$$\mathbf{a} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

are linearly dependent and hence, or otherwise, find the value of z which satisfies the given equations when there is a unique solution.

[JMB, 1970]

2. (a) Prove that the equation

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is soluble only if $c + 2b - 5a = 0$.

(b) Hence show that the planes

$$x + 2y - 3z = 1$$
$$2x + 6y - 11z = 2$$
$$x - 2y + 7z = 1$$

intersect in a line.

(c) Find, in terms of s, the coordinates of the point in which this line meets the plane $z = s$.

[JMB, 1978]

3. (a) Find a vector equation for the line of intersection of the planes

$$2x + y - z = 3$$
$$x + 2y + 4z = 0$$

(b) Hence, or otherwise, solve the equations

$$2x + y - z = 3$$
$$x + 2y + 4z = 0$$
$$3x + \lambda y + 6z = 2$$

for **all** real values of λ, interpreting your results geometrically.

[JMB, 1980]

4. (a) Express the determinant

$$\mathbf{D} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

as the product of four linear factors.

(b) Two points A and B have coordinates (1, 2, 8) and (1, 3, 27), respectively. A third point C, which is distinct from A and B, has coordinates $(1, c, c^3)$. Given that the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ are linearly dependent, find the value of c.

[NEAB]

5. Given that **a**, **b** and **c** are linearly independent vectors, determine if the following vectors are linearly independent:

   (a) $\mathbf{a}, \mathbf{0}$

   (b) $\mathbf{a}+\mathbf{b}, \quad \mathbf{b}+\mathbf{c}, \quad \mathbf{c}+\mathbf{a}$

   (c) $\mathbf{a}+2\mathbf{b}+\mathbf{c}, \quad \mathbf{a}-\mathbf{b}-\mathbf{c}, \quad 5\mathbf{a}+\mathbf{b}-\mathbf{c}$.

Miscellaneous exercises 6

1. The vectors **a**, **b** and **c**, given below, are linearly independent.

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

   Find α, β and γ such that the vector

$$\mathbf{d} = \begin{bmatrix} 7 \\ 5 \\ -14 \end{bmatrix}$$

   can be expressed as a linear combination of **a**, **b** and **c** in the form

$$\mathbf{d} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}.$$

[NEAB]

2. The vector **r** is defined as $\mathbf{r} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$.

   (a) Write down the resolved part of the vector **r** in the direction of **a**, where $\mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

   (b) Find the resolved parts of the vector **r** in the directions **b** and **c**, where

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

   (c) Using the fact that **a**, **b** and **c** are mutually perpendicular, or otherwise, express **r** as a linear combination of **a**, **b** and **c**.

[AQA-NEAB, 2000]

3. (a) Show that the simultaneous equations

$$6x - 7y + 2z = 4$$
$$6x - y - z = 7$$
$$2x - 3y + z = k,$$

where k is a constant, are consistent only when $k = 1$.

(b) Give, with reasons, a geometrical interpretation of the three equations in **each** of the cases when $k = 1$ and $k \neq 1$.

[AQA-NEAB, 2000]

4. The vectors **a**, **b** and **c** are defined by

$$\mathbf{a} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -k \\ k \\ 1 \end{bmatrix},$$

where k is a real number.

(a) Show that the vectors **a**, **b** and **c** are linearly independent for all values of k.

(b) The columns of the matrix **M** are the components of **a**, **b** and **c**, respectively.

(i) Obtain $\mathbf{M}^{-1}$ in terms of k.

(ii) Solve the equations

$$x + y - kz = 5$$
$$5x + 3y + kz = 1$$
$$3x + 2y + z = 2,$$

giving your answers in terms of k.

[AQA-NEAB, 2001]

5. (a) Express the determinant

$$\begin{vmatrix} a & bc & b+c \\ b & ca & c+a \\ c & ab & a+b \end{vmatrix}$$

as the product of four linear factors.

(b) (i) Hence, or otherwise, find the values of a for which the simultaneous equations

$$ax + 2y + 3z = 0$$
$$2x + ay + (1+a)z = 0$$
$$x + 2ay + (2+a)z = 0$$

have a solution other than $x = y = z = 0$.

(ii) Solve the equations when $a = -3$.

[NEAB]

6. (a) (i) Given that

$$\mathbf{a} = \begin{bmatrix} -1 \\ 3 \\ 13 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix},$$

express **a** as a linear combination of **b** and **c**.

(ii) Hence, or otherwise, evaluate

$$\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 15 \\ -1 & 3 & 13 \end{vmatrix}.$$

(b) A transformation of 3-dimensional space is defined by $\mathbf{r}' = \mathbf{M}\mathbf{r}$, where

$$\mathbf{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 5 \\ -1 & 3 & 13 \end{bmatrix}.$$

Show that for all values of **r** the point with position vector $\mathbf{r}'$ lies in a plane, and find an equation for this plane.

[JMB, 1980]

7. (a) Show that the only real value of λ, for which the simultaneous equations

$$(2+\lambda)x - y + z = 0$$
$$x - 2\lambda y - z = 0$$
$$4x - y - (\lambda - 1)z = 0$$

have a solution other than $x = y = z = 0$, is -1.

(b) Solve the equations in the case when $\lambda = -1$, and interpret your result geometrically.

[NEAB]

8. The matrix **A** is given by $\begin{bmatrix} 3 & 3 & a \\ 2 & -1 & 1 \\ 0 & 3 & 1 \end{bmatrix}$, where $a \neq 3$.

(a) Find the inverse of **A**.

(b) Hence, or otherwise, find the point of intersection of the three planes with equations

$$3x + 3y + az = 3 + 2a$$
$$2x - y + z = 7$$
$$3y + z = -1.$$

9. Find the solution set for the system of equations

$$x-2y-z=6$$
$$3x+y+5z=-3$$
$$x+5y+7z=-15$$

and interpret the solution geometrically when the three equations represent the Cartesian equations of three planes.

[AQA]

10. (a) Express the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$ as a product of linear factors.

(b) Under what conditions on a, b and c does the following set of equations have a unique solution?

$$x+y+z=3$$
$$ax+by+cz=1$$
$$a^2x+b^2y+c^2z=5$$

11. The real numbers a, b and c are not all equal, and you can assume that $a^2+b^2+c^2>ab+bc+ca$.
The simultaneous equations

$$ax+by+cz=0$$
$$bx+cy+az=0$$
$$cx+ay+bz=0$$

have a solution other than $x=y=z=0$. Show that $a+b+c=0$.

[JMB, 1973]

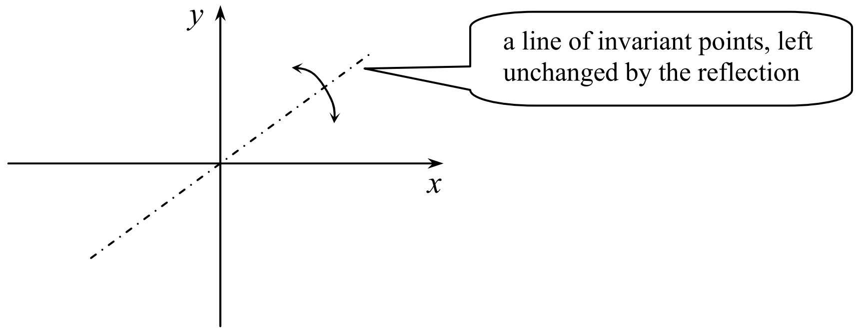
Chapter 7: Eigenvectors

This chapter introduces the idea of studying points and lines which are fixed by matrix transformations. When you have completed it, you will:

- be able to find the invariant points and lines of transformations;
- know what is meant by the eigenvectors and eigenvalues of a matrix transformation;
- understand why the eigenvalues are the roots of the characteristic equation;
- be able to diagonalise square matrices which have a full set of eigenvectors.

7.1 Invariant points and lines

An **invariant point** of a transformation is a point which is unchanged by the transformation. For example, a reflection in the line $y = x$ leaves every point on the line $y = x$ unchanged.



The invariant points of a matrix transformation can be found by solving a matrix equation.

The invariant points of the transformation with matrix **M** can be found by solving $\mathbf{Mx} = \mathbf{x}$

Example 7.1.1

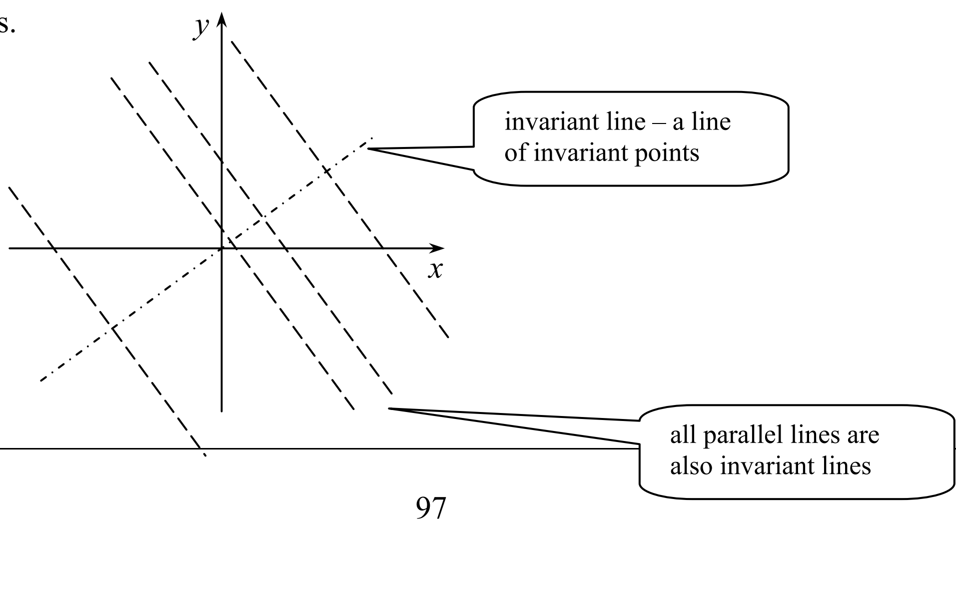
Find the invariant points of the transformation with matrix $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where $k \neq 0$.

Solution

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\Rightarrow kx + y = y$$
$$\Rightarrow x = 0.$$

The invariant points are all the points on the line $x = 0$.

An **invariant line** of a transformation is a line which is left 'fixed' by the transformation. In some cases, this line will consist of invariant points. In others, it will consist of points which are moved around on the line. For example, a reflection in the line $y = x$ has infinitely many invariant lines.



The invariant lines of a matrix transformation can also be found by solving equations.

The invariant lines of the transformation with matrix **M** can be found by substituting **x** and **Mx** in the same equation of a line

Example 7.1.2

Find, in the form $y = mx + c$, the invariant lines of the transformation with matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ mx+c \end{bmatrix} = \begin{bmatrix} mx+c \\ x \end{bmatrix}$$

$$\Rightarrow x = m(mx+c)+c$$

$$\Rightarrow (1-m^2)x = (m+1)c, \quad \text{for all } x.$$

Put $x = 0$, then $(m+1)c = 0 \quad \Rightarrow m = -1$ or $c = 0$.

Then $(1-m^2)x = 0$, for all $x \quad \Rightarrow m = \pm 1$.

So, $\quad m = -1 \quad$ or $\quad m = 1, c = 0$

$\quad \Rightarrow y = -x + c \quad$ or $\quad y = x$.

The invariant points and lines of a transformation are closely connected with its geometrical significance. Thus a rotation in 3-dimensional space always has a line of invariant points – the axis of the rotation – but only has other invariant lines if the rotation is 180°.

For complicated transformations, it can be useful to determine any invariant points and lines before attempting to decide on the nature of the transformation. Section 7.2 develops these ideas further.

Exercise 7A

1. Find all invariant lines, of the form $y = mx$, for the matrix transformations

   (a) $\begin{bmatrix} 5 & 15 \\ -2 & -8 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$.

2. The plane transformation T is defined by

$$x' = 4x + 2y - 7$$
$$y' = -3x - y + 7.$$

   (a) Show that T has a line of invariant points and find its equation.

   (b) Show that there is an infinite number of invariant lines and find the general equation of such lines.

[AQA]

3. (a) Given the matrix $\mathbf{M} = \frac{1}{13}\begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$, evaluate $\mathbf{M}^2$ and the determinant of $\mathbf{M}$.

   (b) Find a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, and also a vector $\begin{bmatrix} u \\ v \end{bmatrix}$ such that $\mathbf{M}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u \\ -v \end{bmatrix}$.

   (c) Describe, in geometrical terms, the transformation represented by the matrix **M**.

[JMB, 1977]

4. A transformation T of 3-dimensional space is defined by $\mathbf{r}' = \mathbf{Mr}$, where

$$\mathbf{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{M} = \begin{bmatrix} 1 & 3 & 2 \\ -3 & 6 & 4 \\ 5 & -3 & -2 \end{bmatrix}.$$

   (a) Prove that T transforms the plane $x = a$ into a line, and show that the direction of this line is independent of the value of a.

   (b) Given that this line passes through the point (5, –5, 13), determine the value of a.

[JMB, 1979]

5. Find a line of invariant points for the transformation with matrix $\mathbf{M} = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 6 & 6 \\ 8 & -6 & -3 \end{bmatrix}$.

7.2 Eigenvectors and eigenvalues

If a square matrix **M** and vector **v** satisfy the equation

$$\mathbf{Mv} = \lambda\mathbf{v}, \qquad \mathbf{v} \neq \mathbf{0},$$

then **v** is called an **eigenvector** of the transformation and λ is called an **eigenvalue**.

If **v** is an eigenvector, then the line $\mathbf{r} = t\mathbf{v}$ is an invariant line through the origin. In particular, any multiple of **v** is itself an eigenvector.

The eigenvectors of a transformation determine the directions of all invariant lines which pass through the origin

A special method has been developed to solve the important equation

$$\mathbf{Mv} = \lambda\mathbf{v}, \qquad \mathbf{v} \neq \mathbf{0}.$$

First, it is written as $(\mathbf{M} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$. If the matrix $\mathbf{M} - \lambda\mathbf{I}$ had an inverse then you could multiply by this inverse, obtaining $\mathbf{v} = \mathbf{0}$. Since $\mathbf{v} \neq \mathbf{0}$, the conclusion must be that $\mathbf{M} - \lambda\mathbf{I}$ is singular.

This leads to the following method for finding eigenvalues:

The eigenvectors for a matrix **M** can be found by solving $\lvert\mathbf{M} - \lambda\mathbf{I}\rvert = 0$

7.3 The characteristic equation

The equation $|\mathbf{M}-\lambda\mathbf{I}|=0$ is called the **characteristic equation** for the matrix **M**.

Example 7.3.1

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M}=\begin{bmatrix}3&2&2\\-1&0&1\\1&2&1\end{bmatrix}$.

$$\begin{vmatrix}3-\lambda&2&2\\-1&-\lambda&1\\1&2&1-\lambda\end{vmatrix}=0$$
$$\Rightarrow (3-\lambda)\left[-\lambda(1-\lambda)-2\right]-2\left[-(1-\lambda)-1\right]+2(-2+\lambda)=0$$
$$\Rightarrow -\lambda^3+4\lambda^2-\lambda-6=0$$
$$\Rightarrow -(\lambda+1)(\lambda-2)(\lambda-3)=0$$
$$\Rightarrow \lambda=-1, 2 \text{ or } 3.$$

The eigenvector for the eigenvalue $\lambda=-1$ is given by

$$\begin{bmatrix}4&2&2\\-1&1&1\\1&2&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{0} \qquad \Rightarrow \begin{bmatrix}3&0&0\\0&3&3\\1&2&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{0} \qquad \text{E.g.}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$
$$\Rightarrow x=0, \quad y+z=0$$

For $\lambda = 2$,

$$\begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \quad \Rightarrow \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \quad \text{E.g.} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow z = 0, \quad x + 2y = 0$$

For $\lambda = 3$,

$$\begin{bmatrix} 0 & 2 & 2 \\ -1 & -3 & 1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \quad \Rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 0 & -1 & -1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \quad \text{E.g.} \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow y = -z, \quad x = 2z - 2y$$

Example 7.3.2 shows a case where the characteristic equation was of order 3 and had precisely three solutions. Sometimes, as in the next two examples, the characteristic equation may have fewer solutions than its order.

Example 7.3.2

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} -8 & -4 \\ 25 & 12 \end{bmatrix}$.

Solution

$$\begin{vmatrix} -8-\lambda & -4 \\ 25 & 12-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-8-\lambda)(12-\lambda) + (4 \times 25) = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2.$$

$$\begin{bmatrix} -10 & -4 \\ 25 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow 5x + 2y = 0$$

E.g. $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$.

This matrix transformation has only one eigenvalue and associated eigenvector.

Example 7.3.3

Find the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} -1 & -2 & 4 \\ -2 & 2 & 2 \\ 4 & 2 & -1 \end{bmatrix}$.

Solution

$$\begin{vmatrix} -1-\lambda & -2 & 4 \\ -2 & 2-\lambda & 2 \\ 4 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(\lambda-3)^2(\lambda+6) = 0$$

$$\Rightarrow \lambda = 3 \text{ or } -6.$$

For $\lambda = -6$,

$$\begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & 9 \\ -18 & 0 & -18 \\ 4 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow x = -z, \quad 4x + 2y + 5z = 0$$

E.g. $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$.

For $\lambda = 3$,

$$\begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow 2x + y = 2z$$

E.g. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

In this case, the eigenvalue 3 has an entire plane of eigenvectors – any linear combination of $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Exercise 7B

1. Show that $\begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $\mathbf{M} = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 3 & 2 \\ 1 & 0 & 5 \end{bmatrix}$ and find the associated eigenvalue.

2. Find the characteristic equation of the matrix $\begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$.

3. (a) Find a line of invariant points for the transformation with matrix
$$\mathbf{M} = \begin{bmatrix} 5 & 5 & 7 \\ -10 & 3 & 5 \\ 18 & 8 & 10 \end{bmatrix}.$$

   (b) Hence find the eigenvalues and eigenvectors for **M**.

4. Find the eigenvalues and corresponding eigenvectors for the matrix $\mathbf{M} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 5 \\ 1 & 1 & -1 \end{bmatrix}$.

5. The matrix **A** is defined by $\mathbf{A} = \begin{bmatrix} a & 0 & -a \\ 0 & 4 & 0 \\ -a & 0 & 5 \end{bmatrix}$, where a is a real number.

   (a) Given that $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector of **A**, find the corresponding eigenvalue λ_1.

   (b) Given that $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is another eigenvector of **A**, show that $a = 2$ and find the eigenvalue λ_2 corresponding to $\mathbf{v}_2$.

   (c) Given that the third eigenvalue λ_3 is 6, find a corresponding eigenvector $\mathbf{v}_3$.

[AQA-NEAB, 2001]

7.4 Diagonalisation

A **diagonal matrix** is a square matrix which only has non-zero elements on the leading diagonal. For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

The geometrical effect of a diagonal matrix can be seen clearly. For example,
$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, with associated eigenvalues 1, 3 and –5, respectively.

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is mapped to itself. The x-axis is a line of invariant points.

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is mapped to 3 times itself. The y-axis is invariant.

$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is mapped to –5 times itself. The z-axis is invariant.

The effect of **M** is that of a one-way stretch, $\times 3$, in the direction of the y-axis and a one-way stretch, $\times -5$, in the direction of the z-axis.

In fact, when a matrix transformation has a full set of eigenvectors (i.e. when an $n \times n$ matrix has n independent eigenvectors) then the matrix is actually closely related to a diagonal matrix with the eigenvalues as its diagonal elements. This idea will first be illustrated with a simple example.

Example 7.4.1

Let $\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix}$.

(a) Find the eigenvalues $\mathbf{v}_1$, $\mathbf{v}_2$ and associated eigenvectors λ_1, λ_2 of **M**.

(b) Let **V** be the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ and **D** the diagonal matrix $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Find $\mathbf{VDV}^{-1}$.

Solution

(a) $\begin{vmatrix} 4-\lambda & 1 \\ 6 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda = 1 \text{ or } 6.$

For $\lambda_1 = 1, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. For $\lambda_2 = 6, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) $\mathbf{V} = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$ and $\mathbf{V}^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$.

$$\mathbf{VDV}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 18 & 6 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 20 & 5 \\ 30 & 15 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix}.$$

In this case, $\mathbf{M} = \mathbf{VDV}^{-1}$.

The process of expressing a matrix in the form $\mathbf{VDV}^{-1}$, where **D** is a diagonal, is called **diagonalisation**.

> An $n \times n$ matrix **M** which has n independent eigenvectors can be expressed as $\mathbf{VDV}^{-1}$, where **V** is an invertible matrix and **D** is a diagonal matrix
>
> **V** consists of the eigenvectors of **M**, and **D** consists of the eigenvalues. The order of the eigenvalues in **D** corresponds to the order of the eigenvectors in **V**

Example 7.4.2

Find an invertible matrix **V** and a diagonal matrix **D** such that $\begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \mathbf{VDV}^{-1}$.

Solution

Using the eigenvalues and eigenvectors found in Example 7.3.1,

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0 & 2 & 4 \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Check: $$\mathbf{VD} = \begin{bmatrix} 0 & 4 & 12 \\ -1 & -2 & -3 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 2 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}\mathbf{V} = \begin{bmatrix} 0 & 4 & 12 \\ -1 & -2 & -3 \\ 1 & 0 & 3 \end{bmatrix}, \text{ as required.}$$

When $\mathbf{M} = \mathbf{VDV}^{-1}$, the properties of **M** and **D** are very similar. For example,

- $|\mathbf{M}| = |\mathbf{VDV}^{-1}| = |\mathbf{V}||\mathbf{D}||\mathbf{V}^{-1}| = |\mathbf{D}|$.
- $\mathbf{M}^n = \left(\mathbf{VDV}^{-1}\right)\left(\mathbf{VDV}^{-1}\right)\ldots\left(\mathbf{VDV}^{-1}\right)$
  $= \mathbf{VD}\left(\mathbf{V}^{-1}\mathbf{V}\right)\mathbf{D}\left(\mathbf{V}^{-1}\mathbf{V}\right)\ldots\left(\mathbf{V}^{-1}\mathbf{V}\right)\mathbf{DV}^{-1}$
  $= \mathbf{VD}^n\mathbf{V}^{-1}$.

In effect, the matrix **M** is itself a diagonal transformation matrix in a space where the base vectors are the eigenvectors instead of the usual vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

If $\mathbf{M} = \mathbf{VDV}^{-1}$, then $\mathbf{M}^n = \mathbf{VD}^n\mathbf{V}^{-1}$.

Example 7.4.3

The matrix **M** has eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, with associated eigenvalues λ_1 and λ_2. Find **Mv**, where $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$.

Solution

$$\begin{aligned}\mathbf{Mv} &= \mathbf{M}\left(a\mathbf{v}_1 + b\mathbf{v}_2\right)\\ &= a\left(\mathbf{Mv}_1\right) + b\left(\mathbf{Mv}_2\right)\\ &= a\left(\lambda_1\mathbf{v}_1\right) + b\left(\lambda_2\mathbf{v}_2\right)\\ &= \lambda_1\left(a\mathbf{v}_1\right) + \lambda_2\left(b\mathbf{v}_2\right).\end{aligned}$$

The idea of expressing a matrix **M** in the form $\mathbf{VDV}^{-1}$ works in general, providing that **M** has a full set of eigenvectors. This can be proved reasonably simply.

Let **M** be a 3×3 matrix with eigenvalues λ_1, λ_2 and λ_3 and associated eigenvalues $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, respectively. The eigenvectors must be independent.

Let **V** be the matrix of eigenvectors $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$.

$\mathbf{Mv}_i = \lambda_i\mathbf{v}_i$ and so $\mathbf{MV} = \begin{bmatrix}\lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \lambda_3\mathbf{v}_3\end{bmatrix}$.

Because the eigenvectors are independent, the matrix **V** is invertible. Then $\mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$ and so

$$\mathbf{V}^{-1}\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{V}^{-1}\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ and } \mathbf{V}^{-1}\mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Then

$$\begin{aligned}\mathbf{V}^{-1}\mathbf{MV} &= \mathbf{V}^{-1}\begin{bmatrix}\lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \lambda_3\mathbf{v}_3\end{bmatrix}\\ \Rightarrow \mathbf{V}^{-1}\mathbf{MV} &= \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\\ \Rightarrow \mathbf{M} &= \mathbf{V}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}\mathbf{V}^{-1}.\end{aligned}$$

Exercise 7C

1. Write down the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \mathbf{VDV}^{-1}$, where

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } \mathbf{V} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & -1 & 1 \end{bmatrix}.$$

2. Express $\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

3. The matrix **M** is of the form $\mathbf{V}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{V}^{-1}$. Find the determinant of **M**.

4. Diagonalise each of the following matrices. [Use your answers to Questions 3, 4 and 5 of Exercise 7B.]

(a) $\begin{bmatrix} 5 & 5 & 7 \\ -10 & 3 & 5 \\ 18 & 8 & 10 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 5 \\ 1 & 1 & -1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$.

Miscellaneous exercises 7

1. A linear transformation T of 3-dimensional space is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

(a) Show that **M** has just two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$.

(b) Find an eigenvector corresponding to λ_1.

(c) Show that all vectors in the plane $x + y + z = 0$ are eigenvectors corresponding to λ_2.

(d) (i) Write down a vector equation of a line L which is invariant under T.

(ii) State, giving a reason, whether points on L are invariant under T.

[AQA-NEAB, 2000]

2. For constants a and b, the matrix $\mathbf{M} = \begin{bmatrix} a & 2 \\ b & 3a \end{bmatrix}$ and the plane transformation $\mathbf{T}$ is such that $\mathbf{T} : \begin{bmatrix} x \\ y \end{bmatrix} \to \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$.

   (a) (i) Determine the eigenvalues of **M**, giving your answers in terms of a and b in their simplest form.

   (ii) In the case when $a = 3$ and $b = -6$, show that **M** has no real eigenvalues.

   (b) A general point on the unit circle, centre the origin, is $P(\cos\theta, \sin\theta)$.

   (i) Show that, with $a = 3$ and $b = -6$, the image of P under **T** can be written in the form $P'\left[R_1 \cos(\theta - \alpha), R_2 \sin(\theta - \alpha)\right]$.

   State the values of the positive real constants R_1 and R_2, and find the exact value of $\tan\alpha$.

   (ii) Describe the locus of P' as θ varies.

[AQA-NEAB, 2000]

3. (a) Find the characteristic equation for the matrix $\mathbf{M} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & -2 \\ 4 & 4 & -3 \end{bmatrix}$.

   (b) Hence find the eigenvalues of **M** and the corresponding eigenvectors.

4. (a) Find the eigenvalues and corresponding eigenvectors of $\mathbf{M} = \begin{bmatrix} 3 & -2 & -2 \\ 2 & -1 & -4 \\ -1 & 1 & 4 \end{bmatrix}$.

   (b) Hence express **M** in the form $\mathbf{VDV}^{-1}$, where **D** is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

5. The matrix $\mathbf{M} = \begin{bmatrix} 2 & 3 & 0 \\ 2 & 4 & 1 \\ 0 & -3 & 2 \end{bmatrix}$.

   (a) Show that **M** has an eigenvector $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. Find the corresponding eigenvalue.

   (b) Hence find two further eigenvalues and corresponding eigenvectors.

   (c) Express **M** in the form $\mathbf{VDV}^{-1}$.

6. (a) Express $\mathbf{M} = \frac{1}{6}\begin{bmatrix} -1 & 9 & -5 \\ 1 & 3 & -1 \\ -4 & 0 & -2 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

(b) Hence find $\mathbf{M}^n$ for any integral n.

7. The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

(a) (i) Find the eigenvalues of $\mathbf{A}$.

(ii) For each eigenvalue find a corresponding eigenvector.

(b) Given that $\mathbf{U} = \begin{bmatrix} a & 5 \\ -3 & b \end{bmatrix}$, write down the values of a and b such that

$\mathbf{U}^{-1}\mathbf{AU} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$. [NEAB]

8. The matrix $\mathbf{A}$ is given by $\begin{bmatrix} 7 & 4 \\ -1 & 3 \end{bmatrix}$. The plane transformation $\mathbf{T}$ is such that

$\mathbf{T}: \begin{bmatrix} x \\ y \end{bmatrix} \to \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) (i) Show that $\mathbf{A}$ has only one eigenvalue. Find this eigenvalue and a corresponding eigenvector.

(ii) Hence, or otherwise, determine a Cartesian equation of the fixed line of $\mathbf{T}$.

(b) Under $\mathbf{T}$, a square with area $1\,\text{cm}^2$ is transformed into a parallelogram with area $d\,\text{cm}^2$. Find the value of d.

[AEB, 1996]

9. Express $\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

10. The matrix **P** is defined by $\mathbf{P} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$.

(a) Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are eigenvectors of **P** and find the two corresponding eigenvalues.

(b) Given that the third eigenvalue of **P** is 4, find the corresponding eigenvector $\mathbf{v}_3$.

(c) Show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent.

(d) Express the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ with coefficients in terms of the constants a, b and c.

[NEAB]

11. Let **A** be the matrix $\begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$.

(a) Determine the eigenvalues and corresponding eigenvectors of **A**.

(b) (i) Show that $\mathbf{A}^2 - 2\mathbf{A} - 8\mathbf{I} = \mathbf{Z}$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{Z} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(ii) The matrix $\mathbf{B} = \mathbf{A}^{-1}$. By multiplying the matrix equation $\mathbf{A}^2 - 2\mathbf{A} - 8\mathbf{I} = \mathbf{Z}$ by **B**, or otherwise, find the values of the scalars α and β for which $\mathbf{B} = \alpha\mathbf{A} + \beta\mathbf{I}$.

[AEB, 1997]

12. The transformation T maps points (x, y) of the plane into image points (x', y') such that

$$x' = 4x + 2y + 14$$
$$y' = 2x + 7y + 42.$$

(a) (i) Find the coordinates of the invariant point of T.

(ii) Hence express T in the form $\begin{bmatrix} x' \\ y' + k \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y + k \end{bmatrix}$, where k is a positive integer and **A** is a 2×2 matrix.

(b) (i) Determine the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$.

(ii) Deduce the Cartesian equations of the invariant lines of T, and prove that they are perpendicular.

(c) Give a full geometrical description of T.

[AEB, 1998]

13. A linear transformation of 3-dimensional space is defined by $\mathbf{r}' = \mathbf{M}\mathbf{r}$, where

$$\mathbf{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{M} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 3 \\ 2 & k & 4 \end{bmatrix}.$$

(a) Show that the transformation is singular if, and only if, $k = 2$.

(b) In the case when $k = 2$, show that **M** represents a transformation of 3-dimensional space onto a plane, and find a Cartesian equation of this plane.

[NEAB]

14. The matrix **A** is defined by $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{bmatrix}$.

(a) Find the determinant of **A** in terms of k.

(b) The matrix **A** corresponds to a linear transformation T in 3-dimensional space. When a region in 3-dimensional space is transformed by T, its volume, V, is increased by a factor of 4 to $4V$. Find the possible values of k.

[NEAB]

15. Determine the eigenvalues and corresponding eigenvectors of the matrix **A**, where

$$\mathbf{A} = \begin{bmatrix} 26 & -5 \\ -5 & 2 \end{bmatrix}.$$

The plane transformation **T** is defined by $\mathbf{T} : \begin{bmatrix} x \\ y \end{bmatrix} \to \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) Write down a Cartesian equation of the line of invariant points of **T**.

(b) Show that all lines of the form $y = -\frac{1}{5}x + k$, where k is an arbitrary constant, are invariant lines of **T**.

(c) Evaluate the determinant of **A** and explain the geometrical significance of this answer in relation to **T**.

(d) Give a full geometrical description of **T**.

[AEB, 1998]

Answers to Exercises - Further Pure 4

Chapter 1

Exercise 1A

1. 2×4 2. 2 3. 2×1 4. $\begin{bmatrix}3 & -1 & 2\\ 4 & 2 & -3\end{bmatrix}$

Exercise 1B

1. $\mathbf{AB}=\begin{bmatrix}2 & -4\\ 3 & -1\end{bmatrix}$; $\mathbf{BA}=\begin{bmatrix}2 & -1 & -1\\ 3 & 1 & -1\\ 7 & 4 & -2\end{bmatrix}$

2. (a) $\mathbf{A}^2=\begin{bmatrix}-3 & -4\\ 4 & -3\end{bmatrix}$; $\mathbf{B}^2=\begin{bmatrix}2 & 10\\ -5 & 7\end{bmatrix}$; $\mathbf{AB}=\begin{bmatrix}4 & -4\\ 3 & 7\end{bmatrix}$; $\mathbf{BA}=\begin{bmatrix}6 & -2\\ 5 & 5\end{bmatrix}$

(b) $\mathbf{A}+\mathbf{B}=\begin{bmatrix}3 & 0\\ 1 & 4\end{bmatrix}$; $(\mathbf{A}+\mathbf{B})^2=\begin{bmatrix}9 & 0\\ 7 & 16\end{bmatrix}$; $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2=\begin{bmatrix}9 & 0\\ 7 & 16\end{bmatrix}$

3. Only the square matrices, **B** and **D**

4. $\mathbf{A}^2=\begin{bmatrix}7 & 1\\ 3 & 4\end{bmatrix}$; $\mathbf{A}+5\mathbf{I}=\begin{bmatrix}7 & 1\\ 3 & 4\end{bmatrix}$

5. $(\mathbf{AB})\mathbf{C}=(9-2)=\mathbf{A}(\mathbf{BC})$

6. **X** has order 3×4; **Y** has order 2×1

7. $\mathbf{A}(\mathbf{B}+\mathbf{C})=\begin{bmatrix}10 & -3 & 10\\ -1 & 0 & -1\end{bmatrix}$; $\mathbf{AB}=\begin{bmatrix}5 & -2 & 5\\ -2 & -1 & 1\end{bmatrix}$; $\mathbf{AC}=\begin{bmatrix}5 & -1 & 5\\ 1 & 1 & -2\end{bmatrix}$

$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$

8. $\begin{bmatrix}7 & 2 & -3\\ 1 & 11 & -4\\ 1 & -1 & -2\end{bmatrix}$

Exercise 1C

1. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or any matrix whose columns are multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2. For example, $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

3. $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ for any a and d

5. $\mathbf{AN} = \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} = \mathbf{MC}$; by associativity

6. $\mathbf{A}(\mathbf{B} - \mathbf{C}) = \mathbf{AB} - \mathbf{AC} = \begin{bmatrix} -2 & 0 \\ -5 & 3 \end{bmatrix}$

7. $(\mathbf{AB})^{\mathrm{T}} = \begin{bmatrix} 11 & 4 \\ 2 & 1 \end{bmatrix} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$

Exercise 1D

1. (a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; rotation of 180° about O

   (b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; rotation of 90° clockwise about O

2. Each squared matrix is the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The result of performing any reflection twice is to return all points to their original positions

3. (a) Identity – all points stay fixed
   (b) Zero – all points map on to the origin
   (c) A rotation of $\text{Atn}\left(\dfrac{4}{3}\right)$ and an enlargement of $\times 5$, both about the origin

4. $x' = a^2x + aby, \quad y' = abx + b^2y \quad \Rightarrow y' = \dfrac{b}{a}x'.$

5. (a) $\dfrac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ (b) $\dfrac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{bmatrix}$;
   reflection in the line $y = -x\tan\theta$

7. $\begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2(\phi-\theta) & -\sin 2(\phi-\theta) \\ \sin 2(\phi-\theta) & \cos 2(\phi-\theta) \end{bmatrix}$;
   rotation of $2(\phi-\theta)$, anticlockwise

8. $\begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

Miscellaneous exercises 1

1. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ (b) $\cos\theta = \frac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}; \quad \theta = \frac{\pi}{3}$ (c) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \end{bmatrix}$

2. $\begin{bmatrix} 4a+b & 2a+3b \\ 4c+d & 2c+3d \end{bmatrix} = \begin{bmatrix} 4a+2c & 4b+2d \\ a+3c & b+3d \end{bmatrix} \Rightarrow b = 2c, \quad a = c+d$

3. (a) The line $x = y = z$ (b) 120°

4. (a) $y = \frac{x}{2}$ (b) $m = \pm 2$

5. (a) $y = -\frac{4x}{5}$ (b) $y = 2x; \quad y = -\frac{x}{2}$

6. (a) Rotation through θ about O anticlockwise; reflection in $y = x\tan\phi$
   (b) Reflection in $y = x\tan q$ followed by reflection in $y = x\tan p$ is equivalent to rotation through $2(p-q)$; rotation through q followed by reflection in $y = x\tan p$ is equivalent to reflection in $y = x\tan p$ followed by rotation through $-q$

7. (a) T_1: a rotation through 90° about the x-axis such that the positive y-axis maps on to the positive z-axis
   (b) T_2: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; T_3: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$
   (c) T_3T_1: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; T_1T_3: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
   T_3T_1 causes a rotation of 180° about the z-axis; T_1T_3 causes a rotation of 180° about the y-axis

Chapter 2

Exercise 2A

1. The results are summarised at the start of Section 2.2

Exercise 2B

1. $-\mathbf{j}+\mathbf{k}$
2. $\mathbf{j}-\mathbf{k}$
3. $2\mathbf{i}-6\mathbf{j}$
4. $\mathbf{i}-\mathbf{j}-\mathbf{k}$
5. $-3\mathbf{i}+6\mathbf{j}-3\mathbf{k}$
6. $-18\mathbf{i}+3\mathbf{j}+11\mathbf{k}$
7. $\begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}$
8. $\begin{bmatrix} -6 \\ -3 \\ 4 \end{bmatrix}$

Exercise 2C

1. $\frac{1}{2}\sqrt{185}$

2. (a) 1 (b) 3 (c) 0

3. For example, $\mathbf{i}\times(\mathbf{i}\times\mathbf{j})=\mathbf{i}\times\mathbf{k}=-\mathbf{j}$

   $(\mathbf{i}\times\mathbf{i})\times\mathbf{j}=\mathbf{0}\times\mathbf{j}=\mathbf{0}$

4. (a)(i) 2, 2 (ii) –4, – 4 (iii) 1, 1

   (b) It *looks* as if $\mathbf{a.b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b.c}$; a proof of this result is given in Section 2.6

Exercise 2D

1. $(\mathbf{a}+\mathbf{b})\times\mathbf{c.r}=(\mathbf{a}+\mathbf{b})\mathbf{.c}\times\mathbf{r}$

   $=\mathbf{a.c}\times\mathbf{r}+\mathbf{b.c}\times\mathbf{r}$

   $=\mathbf{a}\times\mathbf{c.r}+\mathbf{b}\times\mathbf{c.r}$

   $(\mathbf{a}+\mathbf{b})\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{c}$ is perpendicular to any vector $\mathbf{r}$ and is therefore zero, as required

Miscellaneous exercises 2

1. (a) $\mathbf{b}-\mathbf{a}$ is $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$; $\mathbf{c}-\mathbf{a}$ is $-2\mathbf{i}+3\mathbf{j}-6\mathbf{k}$

   (i) -13 (ii) $-12\mathbf{i}+8\mathbf{j}+8\mathbf{k}$

   (b)(i) $\dfrac{-13}{21}$ (ii) $2\sqrt{17}$ (iii) $-3x+2y+2z=3$

2. (a) $\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}=-2\mathbf{a}\times\mathbf{b}$ (b) $\mathbf{a}\times\mathbf{b}=\mathbf{0}$; 90° or 270°

3. $\mathbf{a}\times\mathbf{a}+2\mathbf{a}\times\mathbf{b}+3\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+2\mathbf{b}\times\mathbf{b}+3\mathbf{b}\times\mathbf{c}=\mathbf{i}+\mathbf{3j}-\mathbf{3k}$

Chapter 3

Exercise 3A

1. (a)(i) $6\mathbf{i}$ (ii) $-6\mathbf{i}$ (iii) $10\mathbf{j}$ (iv) $-15\mathbf{k}$

   (b)(i) $3\mathbf{j}\times 2\mathbf{k}$ (ii) $2\mathbf{k}\times 3\mathbf{j}$ (iii) $2\mathbf{k}\times 5\mathbf{i}$ (iv) $3\mathbf{j}\times 5\mathbf{i}$

Exercise 3B

1. (a) –7 (b) 4 (c) 10

2. Only (a)

3. (a) Area is multiplied by 1, then by 4, then by –1. So $|\mathbf{M}| = -4$

   (b) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} \Rightarrow |\mathbf{M}| = -4$

4. (a) $|\mathbf{A}| = -5$; $|\mathbf{B}| = 3$

   (b) $|\mathbf{AB}| = \begin{vmatrix} 5 & 4 \\ 5 & 1 \end{vmatrix} = -15$; $|\mathbf{BA}| = \begin{vmatrix} 8 & 1 \\ -1 & -2 \end{vmatrix} = -15$

   It looks as if $|\mathbf{AB}| = |\mathbf{BA}| = |\mathbf{A}|\times|\mathbf{B}|$

Exercise 3C

1. (a) 3 (b) 30 (c) –1

2. $3\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = (3\times -10) + 7 = -23$

3. (a) 6 (b) 6 (c) $|\mathbf{M}^{\mathrm{T}}| = |\mathbf{M}|$

4. (a) $|\mathbf{A}| = -3$; $|\mathbf{B}| = -7$; $|\mathbf{AB}| = 21$ (b) $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$

Exercise 3D

1. (a) –2 (b) –2

2. $2abc(a-b)(b-c)(c-a)$

3. (a) Row 1 = Row 2 + Row 3

   (b) Subtracting (Row 2 + Row 3) from Row 1 does not change the determinant

Miscellaneous exercises 3

1. (a) 41 (b) $-2p^2$ (c) $-pqr$

2. $(a+b+c)(a-b)(b-c)(c-a)$

3. (a) $(a-b)(a-c)(b-c)(a+b+c+1)$ (b) -1

4. For example,
   - subtract row 3 from row 1 and from row 2
   - subtract column 3 from column 1, and 2 × column 3 from column 2
   - determinant = –11

5. (a) $(a+b+c)(b-a)(c-b)(a-c)$ (b) $abc(a-b)(b-c)(c-a)$

6. The determinant equals $(ab+bc+ca)(a-b)(b-c)(c-a)$ and so $ab+bc+ca=0$

7. $x=\pm 1$

Chapter 4

Exercise 4A

1. (a) $\begin{bmatrix} -10 \\ 11 \\ 13 \end{bmatrix}$ (b) $2\mathbf{i}-\mathbf{j}$

2. (a) 10 (b) 105

3. $\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0}$

$\Rightarrow \mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$

$\Rightarrow \not{a}b\sin(180-C)=\not{a}c\sin(180-B)$

$\Rightarrow b\sin C=c\sin B$

$\Rightarrow \dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

4. Yes

5. $\begin{vmatrix} 2 & 1 & 3 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{vmatrix}=15$

6. $\dfrac{1}{6}\begin{vmatrix} -1 & -2 & -1 \\ 0 & -3 & 1 \\ 1 & 0 & -4 \end{vmatrix}=\dfrac{17}{6}$

Exercise 4B

1. $\mathbf{r} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

2. $\frac{1}{6}\sqrt{30}$

3. $\left(\mathbf{r} - \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right) \times \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = \mathbf{0}$

4. $-x + y + z = 2$

5. $x + 11y - 8z = -1$

6. (a) $3\mathbf{i} + 9\mathbf{j} + 3\mathbf{k}$ (b) $\frac{15}{11}\sqrt{11}$

7. (a) The normal is $\begin{bmatrix} 3 \\ -2 \\ -8 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 8\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} . \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 5$ and $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} . \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} = 5,$ therefore the lines are coplanar

(b) $2x - y + z = 5$

(c) $\mathbf{r} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 11 \\ 3 \end{bmatrix}$

Miscellaneous exercises 4

1. $\mathbf{x} = \lambda\mathbf{a}$

2. (a) Plane through A perpendicular to **b**
   (b) Line through A parallel to **b**

3. $\frac{5}{3}\sqrt{66}$

4. (b)(i) $(\mathbf{r}-\mathbf{c}).\mathbf{b} = 0$
   (ii) $(\mathbf{r}-\mathbf{a}).[(\mathbf{c}-\mathbf{a})\times\mathbf{b}] = 0$ or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu(\mathbf{c}-\mathbf{a})$ or equivalent

6. $\mathbf{r} = (4+t)\mathbf{i} + (6+2t)\mathbf{j} + (4+5t)\mathbf{k}$

7. (a) $\begin{bmatrix} -8 \\ -10 \\ -4 \end{bmatrix}$ (b)(i) $4x+5y+2z=0$ (ii) $3\sqrt{5}$ (c) $x-2z=5$

8. (b)(i) $2x-y+4z=7$ (ii) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{4}{3}\right)$ (iii) $\frac{7}{6}$

9. (a)(i) $\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ (ii) $(-1, -1, 2)$

   (b)(i) $4x+y+16z=27$ (ii) $(-4, -5, 3)$ (iii) $\mathbf{r} = \begin{bmatrix} -6 \\ 3 \\ 3 \end{bmatrix} + t\begin{bmatrix} 2 \\ -8 \\ 0 \end{bmatrix}$

10. (a)(i) $\sin^{-1}\sqrt{\frac{2}{3}}$ (ii) $\frac{1}{12}$ (b)(i) Volume $OABC = \frac{1}{4}$ (ii) 3:1

11. (a)(i) $\begin{bmatrix} 4 \\ -3 \\ -4 \end{bmatrix} \times \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix} = -24\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (ii) $x+z=4$

    (b) $2\sqrt{2}$ (c) $\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}.\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 5\sqrt{2}\cos(180-\theta) \Rightarrow \theta = 115^\circ$

12. (a) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 14 \\ -5 \end{bmatrix} = 9\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \Rightarrow a = 1, \quad b = 2$

(b)(i) $\overrightarrow{PQ}\,.\mathbf{n} = m\,\mathbf{n}.\mathbf{n}$ (ii) $\sqrt{6}$ units

$\begin{bmatrix} 2 \\ 28 \\ -10 \end{bmatrix} . \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 6m \quad \Rightarrow m = 1$

13. (a) $\overrightarrow{BC} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

(b)(i) $\begin{bmatrix} -2 \\ -1 \\ -2 \end{bmatrix} . \begin{bmatrix} -2 \\ -3 \\ -6 \end{bmatrix} = 21\cos\theta \quad \Rightarrow \cos\theta = \dfrac{19}{21}$ (ii) $\sin\theta = \dfrac{4\sqrt{5}}{21}$

(c) $|AC|\sin\theta = \dfrac{4\sqrt{5}}{3}$

14. (a) $\begin{bmatrix} -6 \\ 3 \\ 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 3 \\ -10 \end{bmatrix} = -9\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ (b) $4x + 6y + 3z = 8$

Chapter 5

Exercise 5A

1. (a) $\frac{1}{7}\begin{bmatrix} 7 & 0 \\ -5 & 1 \end{bmatrix}$ (b) $\frac{1}{7}\begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}$ (c) $-\frac{1}{10}\begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix}$

2. (a) $\frac{1}{6}\begin{bmatrix} 6 & -3 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $-\frac{1}{3}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix}$ (c) $\frac{1}{117}\begin{bmatrix} 9 & 18 & -18 \\ 1 & 15 & 11 \\ 23 & -6 & 19 \end{bmatrix}$

3. (b), (e) and (f)

4. $k = 21$

5. $\frac{1}{2k-5}\begin{bmatrix} -2 & 2 & 3 \\ -1 & 1 & k-1 \\ 2k & -5 & -3k \end{bmatrix}$

6. (a) $-\frac{1}{6}\begin{bmatrix} 4 & -3 & -5 \\ 2 & -3 & -1 \\ -6 & 3 & 3 \end{bmatrix}$ (b) $\mathbf{A}^{-1}\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Miscellaneous exercises 5

1. (a) $|\mathbf{A}| = a^3 + b^3$ (b) $\dfrac{1}{a^3+b^3}\begin{bmatrix} a^2 & -ab & b^2 \\ b^2 & a^2 & -ab \\ -ab & b^2 & a^2 \end{bmatrix}$

   $|\mathbf{A}| \neq 0 \quad \Rightarrow a \neq -b$

2. $\dfrac{1}{4}\begin{bmatrix} -2 & -1 & 4 \\ 4 & 2 & -4 \\ 2 & 3 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

3. $-1, 0, 1$

4. (a) For example, $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

   (b) $\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}(\mathbf{AC})$

   $\Rightarrow \mathbf{IB} = \mathbf{IC}$ (associativity)

   $\Rightarrow \mathbf{B} = \mathbf{C}$

5. (a) $\mathbf{a}$; $\mathbf{b}$

   (b) $\mathbf{a} = (\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}) - 2\mathbf{b} - 3\mathbf{c}$

   $\mathbf{M}^{-1}\mathbf{a} = \mathbf{c} - 2\mathbf{a} - 3\mathbf{b}$

6. (a) $|\mathbf{A}| = k^2 - 8k + 17 \neq 0$ because the discriminant is $-4 < 0$

   (b) $\dfrac{1}{k^2-8k+17}\begin{bmatrix} k-4 & -1 & 20-4k \\ 1 & k-4 & 12-4k \\ 3-k & 5-k & k^2-15 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -4 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

7. $\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} 0 & 1 & -3 \\ 8 & 1 & 9 \\ 5 & 4 & -1 \end{bmatrix}$

8. (a) $\mathbf{A}^2 = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 4 & -6 \\ 0 & 0 & 16 \end{bmatrix}$; $\mathbf{A}^3 = \begin{bmatrix} 1 & 7 & 35 \\ 0 & 8 & -28 \\ 0 & 0 & 64 \end{bmatrix}$

   Then $7\mathbf{A}^2 - 14\mathbf{A} + 8\mathbf{I} = \begin{bmatrix} 1 & 7 & 35 \\ 0 & 8 & -28 \\ 0 & 0 & 64 \end{bmatrix}$, as required

   (b) $\mathbf{A}^3 - 7\mathbf{A}^2 + 14\mathbf{A} = 8\mathbf{I}$

   $\Rightarrow \frac{1}{8}(\mathbf{A}^2 - 7\mathbf{A} + 14\mathbf{I})\mathbf{A} = \mathbf{I}$

Chapter 6

Exercise 6A

1. (a) $\begin{vmatrix} 1 & \lambda & 1 \\ 1 & 1 & \lambda \\ \lambda & 1 & \lambda \end{vmatrix} = (\lambda - 1)^2 (\lambda + 1) \neq 0$ if $\lambda \neq \pm 1$

   (b) $a = b = c$

   (c) $x = a + b, \quad y = b - a + t, \quad z = t$

   (d)(i) Common line (two planes the same) (ii) Two planes parallel

2. (a) $\dfrac{1}{12}\begin{bmatrix} -1 & 5 & 17 \\ 3 & -3 & -15 \\ 4 & -8 & -20 \end{bmatrix}$ (b) $x = 3, \quad y = -2, \quad z = -4$

3. $\lambda = 1$

4. (a) $1 - k^2$ (b) $k \neq \pm 1$ (c) $x = 2, \quad y = 0, \quad z = 1$

Exercise 6B

1. (a) $p = -2$ (b) $\mathbf{a} = \mathbf{b} - 2\mathbf{c}$; $z = 0$

2. (c) $\left(1 - 2s, \ \dfrac{5}{2}s, \ s\right)$

3. (a) $\mathbf{r} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

   (b) $\lambda = 4$: $x = 2 + 2t, y = -1 - 3t, z = t$; a common line (sheaf)

   $\lambda \neq 4$: $x = \dfrac{4}{3}, y = 0, z = -\dfrac{1}{3}$; a unique point

4. (a) $(a - b)(b - c)(c - a)(a + b + c)$ (b) $c = -5$

5. (a) Dependent: $\mathbf{0} = 0 \times \mathbf{a}$ (b) Independent

   (c) Dependent: $5\mathbf{a} + \mathbf{b} - \mathbf{c} = 2(\mathbf{a} + 2\mathbf{b} + \mathbf{c}) + 3(\mathbf{a} - \mathbf{b} - \mathbf{c})$

Miscellaneous exercises 6

1. $\alpha = 2;\ \beta = -3;\ \gamma = 5$

2. (a) -5 (b) $\frac{10}{\sqrt{5}}, \frac{5}{\sqrt{5}}$ (c) $\mathbf{r} = -5\mathbf{a} + 2\mathbf{b} + \mathbf{c}$

3. (a) $9(2x - 3y + z = k) + (6x - y - z = 7) - 4(6x - 7y + 2z = 4)$
 $\Rightarrow 0 = 9(k-1) \quad \Rightarrow k = 1$
 (b) $k = 1$: there are infinitely many solutions; sheaf
 $k \neq 1$: there are no solutions; prism

4. (a) $\begin{vmatrix} 1 & 1 & -k \\ 5 & 3 & k \\ 3 & 2 & 1 \end{vmatrix} = -2 \neq 0$
 (b)(i) $-\frac{1}{2}\begin{bmatrix} 3-2k & -1-2k & 4k \\ -5+3k & 1+3k & -6k \\ 1 & 1 & -2 \end{bmatrix}$ (ii) $x = 2k - 7;\quad y = 12 - 3k;\quad z = -1$

5. (a) $(a-b)(b-c)(c-a)(a+b+c)$ (b) $1, 2, -3$ (c) $\mathbf{r} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

6. (a)(i) $\mathbf{a} = 5\mathbf{c} - 6\mathbf{b}$ (ii) Det $= 0$ (b) $z = 5y - 6x$

7. (a) $\begin{vmatrix} 2+\lambda & -1 & 1 \\ 1 & -2\lambda & -1 \\ 4 & -1 & -(\lambda - 1) \end{vmatrix} = 2(\lambda + 1)(\lambda^2 + 1)$ (b) $\mathbf{r} = t\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$: common line; sheaf

8. (a) $\frac{1}{6a - 18}\begin{bmatrix} -4 & 3a-3 & 3+a \\ -2 & 3 & 2a-3 \\ 6 & -9 & -9 \end{bmatrix}$ (b) $x = 2;\quad y = -1;\quad z = 2$

9. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} 9 \\ 8 \\ -7 \end{bmatrix}$: sheaf

10. (a) $(a-b)(b-c)(c-a)$ (b) a, b and c are all unequal.

11. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(ab + ac + bc - a^2 - b^2 - c^2) \quad \Rightarrow a + b + c = 0$

Chapter 7

Exercise 7A

1. (a) $y = -\frac{1}{5}x;\quad y = -\frac{2}{3}x$ (b) $y = x;\quad y = -\frac{4}{5}x$

2. (a) $3x + 2y = 7$ (b) $y = -\frac{3}{2}x + \frac{7}{2}$ or $y = c - x$

3. (a) $\mathbf{M}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\det = -1$ (b) For example, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

   (c) Reflection in $y = x\tan\frac{\alpha}{2}$, where $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$

4. (a) Direction $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (b) $a = 3$

5. $\mathbf{r} = \lambda \begin{bmatrix} 2 \\ 7 \\ -\frac{13}{2} \end{bmatrix}$

Exercise 7B

1. $\lambda = 2$

2. $\lambda(\lambda + 3)^2 = 0$

3. (a) $\mathbf{r} = \lambda \begin{bmatrix} 11 \\ -90 \\ 58 \end{bmatrix}$ (b) $1, \begin{bmatrix} 11 \\ -90 \\ 58 \end{bmatrix}$; $-2, \begin{bmatrix} 2 \\ 21 \\ -17 \end{bmatrix}$; $19, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

4. $1, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $6, \begin{bmatrix} 16 \\ 19 \\ 5 \end{bmatrix}$; $-2, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

5. (a) 4 (b) 1 (c) $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Exercise 7C

1. $2, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}; \quad -1, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}; \quad 5, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2. $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}$

3. 6

4. (a) $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 19 \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} 11 & 2 & 1 \\ -90 & 21 & 0 \\ 58 & -17 & 2 \end{bmatrix}$

(b) $\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} 1 & 16 & 0 \\ -1 & 19 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

(c) $\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}; \quad \mathbf{V} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

Miscellaneous exercises 7

1. (a) $|\mathbf{M}-\lambda\mathbf{I}| = (2-\lambda)\left(\frac{1}{2}-\lambda\right)^2$ (b) $\begin{bmatrix}1\\1\\1\end{bmatrix}$

   (d)(i) $\mathbf{r} = \lambda\begin{bmatrix}1\\1\\1\end{bmatrix}$ (ii) No; eigenvalue $\neq 1$

2. (a)(i) $2a \pm \sqrt{a^2+2b}$ (ii) a^2+2b is negative

   (b)(i) $\begin{bmatrix}3 & 2\\-6 & 9\end{bmatrix}\begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix} = \begin{bmatrix}3\cos\theta+2\sin\theta\\-6\cos\theta+9\sin\theta\end{bmatrix} = \begin{bmatrix}\sqrt{13}\cos(\theta-\alpha)\\3\sqrt{13}\sin(\theta-\alpha)\end{bmatrix}$; $\tan a = \frac{2}{3}$

   (ii) An ellipse

3. (a) $(\lambda-1)(\lambda+1)(\lambda+2) = 0$ (b) $1, \begin{bmatrix}1\\0\\1\end{bmatrix}$; $-1, \begin{bmatrix}3\\-2\\2\end{bmatrix}$; $-2, \begin{bmatrix}1\\-1\\0\end{bmatrix}$

4. (a) $1, \begin{bmatrix}1\\1\\0\end{bmatrix}$; $2, \begin{bmatrix}2\\0\\1\end{bmatrix}$; $3, \begin{bmatrix}0\\-1\\1\end{bmatrix}$ (b) $\mathbf{V} = \begin{bmatrix}1 & 2 & 0\\1 & 0 & -1\\0 & 1 & 1\end{bmatrix}$

5. (a) 2 (b) $1, \begin{bmatrix}-3\\1\\3\end{bmatrix}$; $5, \begin{bmatrix}1\\1\\-1\end{bmatrix}$ (c) $\mathbf{D} = \begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 5\end{bmatrix}$; $\mathbf{V} = \begin{bmatrix}-3 & 1 & 1\\1 & 0 & 1\\3 & -2 & -1\end{bmatrix}$

6. (a) $\mathbf{V} = \begin{bmatrix}2 & 1 & -1\\1 & 0 & 1\\-1 & 1 & 2\end{bmatrix}$; $\mathbf{D} = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}$

   (b) $\mathbf{M}^n = \mathbf{V}\mathbf{D}^n\mathbf{V}^{-1}$: $\mathbf{D}^n = \begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{bmatrix}$ for n odd; $\mathbf{D}^n = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ for n even

   Hence $\mathbf{M}^n = \mathbf{M}$ for n odd

   and $\mathbf{M}^n = \frac{1}{6}\begin{bmatrix}5 & 3 & -1\\1 & 3 & -1\\2 & -6 & 4\end{bmatrix}$ for n even

7. (a)(i) $-1, \quad 5$ (ii) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $a = 6, \quad b = 5$

8. (a)(i) $(\lambda - 5)^2 = 0 \quad \Rightarrow \lambda = 5: \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (ii) $y = -\frac{1}{2}x$ (b) 25

9. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$

10. (a) $-2, \quad 1$ (b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{vmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0$

(d) $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{a - 2b + c}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{a + b - 2c}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{a + b + c}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

11. (a) $-2, \begin{bmatrix} 1 \\ -5 \end{bmatrix}; \quad 4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (b) $\mathbf{A} - 2\mathbf{I} - 8\mathbf{B} = \mathbf{0}$

$\Rightarrow \mathbf{B} = \frac{1}{8}\mathbf{A} - \frac{1}{4}\mathbf{I}$

12. (a)(i) $(0, -7)$ (ii) $\begin{bmatrix} x' \\ y' + 7 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x' \\ y' + 7 \end{bmatrix}$

(b)(i) $3, \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \quad 8, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (ii) $X + 2Y + 14 = 0$: gradient $-\frac{1}{2}$

$y = 2x - 7$: gradient 2

(c) Two-way stretch from $(0, -7)$: $\times 3$ in the direction of $y = -\frac{1}{2}x$; and $\times 8$ in the direction of $y = 2x$.

13. (a) $|\mathbf{M}| = 10 - 5k \quad \Rightarrow |\mathbf{M}| = 0 \quad \Rightarrow k = 2$ (b) $2x + 2y - z = 0$

14. (a) $k^2 - 2k + 1$ (b) $(k - 1)^2 = 4 \quad \Rightarrow k = -1, 3$

15. (a) $y = 5x$ (b) $\begin{bmatrix} 26 & -5 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} x \\ -\frac{1}{5}x + k \end{bmatrix} = \begin{bmatrix} 27x - 5k \\ -\frac{27}{5}x + 2k \end{bmatrix}$. Then $y' = -\frac{1}{5}x' + k$ as required

(c) $|\mathbf{A}| = 27$. **T** transforms areas by a scale factor of 27. **T** does **not** involve a reflection

(d) A one-way stretch from $y = 5x$